Stat 217, Spring 2023 *Jiaze Qiu* (jiazeqiu@g.harvard.edu)

Section 9: Detection v.s. Recovery [a]

- Sections: Wed, 7:30-8:30pm (SC 705); OHs: Wed 8:30-9:30pm (SC 316.07).
- All the section materials (handouts & solutions) can be found either on Canvas or here.

[a] This handout is based on [9].

Definition 1. *Let distributions P_n, Q_n be defined on the measurable space $(\Omega_n, \mathcal{F}_n)$. We say that the sequence Q_n is contiguous to P_n, and write $Q_n \triangleleft P_n$, if for any sequence A_n of events,*

$$\lim_{n\to\infty} P_n(A_n) = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} Q_n(A_n) = 0.$$

Lemma 1. *If $Q_n \triangleleft P_n$, then there is no hypothesis test of the alternative Q_n against the null P_n with* $\Pr[\textit{type I error}] + \Pr[\textit{type II error}] = o(1)$.

Note that $Q_n \triangleleft P_n$ and $P_n \triangleleft Q_n$ are not equivalent, but either of them implies non-distinguishability.

Our goal today is to show thresholds below which spiked and unspiked random matrix models are contiguous.

Lemma 2. *Let $\{P_n\}$ and $\{Q_n\}$ be two sequences of distributions on $(\Omega_n, \mathcal{F}_n)$. If the second moment*

$$\mathop{\mathbb{E}}_{P_n}\left[\left(\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right)^2\right]$$

exists and remains bounded as $n \to \infty$, then $Q_n \triangleleft P_n$.

1. Prove this lemma.

Lemma 3. *Consider a hypothesis test of a simple alternative Q against a simple null P. Let α be the probability of type I error, and β the probability of type II error. Regardless of the test, we must have*

$$\frac{(1-\beta)^2}{\alpha} + \frac{\beta^2}{(1-\alpha)} \le \mathop{\mathbb{E}}_{P}\left(\frac{\mathrm{d}Q}{\mathrm{d}P}\right)^2,$$

assuming the right-hand side is defined and finite. Furthermore, this bound is tight: for any $\alpha, \beta \in (0,1)$ there exist P, Q and a test for which equality holds.

2. Prove the lemma above and discuss the difference between Lemma 2 and Lemma 3.

Definition 2 (Gaussian Wigner Spiked Matrix Model)**.** *We observe $Y = \lambda x x + \frac{1}{\sqrt{n}} W$, where W is an $n \times n$ random symmetric matrix with entries drawn iid (up to symmetry) from a fixed distribution of mean 0 and variance 1.*

Question 1. *Can we "detect" whether there is a spike or not?*

3. Try to formalize the question above. Is there a difference between "detection" and "recovery"?

We will adopt a Bayesian point of view from now on. Namely, we assume a priori $x \sim \mathcal{X}$, where $\mathcal{X} = \mathcal{X}_n$ is a sequence of distributions on $\mathbb{R}^n$, with the default example being $\mathcal{N}(0, I_n/n)$. It is understood that $\|x\| \approx 1$.

Lemma 4. *Let $\lambda \geq 0$. Let $Q_n = \mathrm{GWig}_n(\lambda, \mathcal{X})$ and $P_n = \mathrm{GWig}_n(0)$. Let x and x' be independently drawn from $\mathcal{X}_n$. Then*

$$\mathop{\mathbb{E}}_{P_n} \left(\frac{\mathrm{d}Q_n}{\mathrm{d}P_n} \right)^2 = \mathop{\mathbb{E}}_{x,x'} \exp\left(\frac{n\lambda^2}{2} \langle x, x' \rangle^2 \right)$$

4. Prove Lemma 4.

It is well known that our spiked Wigner model admits the following spectral behavior.

Theorem 1. *Let Y be drawn from* $\mathrm{GWig}(\lambda, \mathcal{X})$ *with any spike prior $\mathcal{X}$ supported on unit vectors ($\|x\| = 1$):*

- *If $\lambda \leq 1$, the top eigenvalue of Y converges almost surely to 2 as $n \to \infty$, and the top (unit-norm) eigenvector v has trivial correlation with the spike: $\langle v, x \rangle^2 \to 0$ almost surely.*
- *If $\lambda > 1$, the top eigenvalue converges almost surely to $\lambda + 1/\lambda > 2$, and v estimates the spike nontrivially: $\langle v, x \rangle^2 \to 1 - 1/\lambda^2$ almost surely.*

5. Prove that for $\lambda < 1$ "detection" is impossible, assuming $x_i \overset{iid}{\sim} \mathcal{N}(0, 1/n)$.

6. Compare this result to Theorem 1. Do the thresholds for detection adn recovery match? What about more general noise distributions and more general priors on x?

References

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