Stat 217, Spring 2023

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Section 8: Universality and Lindeberg's approach[a]

- Sections: Wed, 7:30-8:30pm (SC 705); OHs: Wed 8:30-9:30pm (SC 316.07).
- All the section materials (handouts & solutions) can be found either on Canvas or here.

[a] This handout is partially based on [3].

Theorem 1. *For $h : \mathbb{R}^n \to \mathbb{R}$ which is thrice differentiable in each coordinate, and $1 \leq r \leq 3$, assume*

$$\lambda_r(h) := \sup_{\mathbf{x},i} |\partial_i^r f(\mathbf{x})| \leq L_r < \infty$$

where ∂_i^r denotes r-fold differentiation with respect to the i^{th} coordinate. Let $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ be two independent families of random variables. Let

$$u_i := |\mathbb{E}a_i - \mathbb{E}b_i|,$$
$$v_i := \left|\mathbb{E}a_i^2 - \mathbb{E}b_i^2\right|.$$

In addition, suppose

$$\max_{1\leq i\leq n} \mathbb{E}\left[|a_i|^3 + |b_i|^3\right] \leq M < \infty.$$

Then

$$|\mathbb{E}h(a) - \mathbb{E}h(b)| \leq L_1 \sum_i u_i + \frac{L_2}{2} \sum_i v_i + \frac{nL_3}{6} M.$$

1. Prove the classical CLT using the theorem above.

2. Prove the above theorem.

3. The Sherrington-Kirkpatrick (S-K) model, can be briefly described as follows: For each $N \geq 1$ let $\left\{J_{ij}^N, 1 \leq i, j \leq N\right\}$ be a collection of i.i.d. $\mathcal{N}(0, 1/N)$ random variables. The S-K model assigns a random probability distribution (the Gibbs measure) on J_N as follows: For any configuration $\sigma \in \Sigma_N = \{-1, 1\}^{\otimes N}$, the probability of the system being in the state $\sigma = (\sigma_1, \ldots, \sigma_N)$ is given by

$$p_{N,J}(\boldsymbol{\sigma}) = Z_{N,J}^{-1} \exp\left(-\beta H_{N,J}(\boldsymbol{\sigma})\right),$$

where $H_{N,g}(\boldsymbol{\sigma}) = -\frac{1}{\sqrt{N}} \sum_{i<j} J_{ij}^N \sigma_i \sigma_j - h \sum_{i \leq N} \sigma_i$, β and h are fixed parameters. It has been shown by [6] that the limit

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left(\log Z_{N,J}\right)$$

exists for all β and h. Then [9] proves (in particular) that

$$\frac{1}{N}\left(\log Z_{N,J} - \mathbb{E} \log Z_{N,J}\right) \xrightarrow{P} 0$$

for any β and h. Both the above facts were proved under the condition that J_{ij}^N are i.i.d. $\mathcal{N}(0, 1/N)$. In fact, the rigorous proofs involve the use of intricate properties of Gaussian random variables.

Now, please try to use the Lindeberg's approach, i.e. Theorem 1, to derive a sufficient condition for invariance of the limiting free energy, with respect to the distributions of entries of J.

References

[1] Mohsen Bayati **and** Andrea Montanari. "The dynamics of message passing on dense graphs, with applications to compressed sensing". **in** *IEEE Transactions on Information Theory*: 57.2 (2011), **pages** 764–785.

[2] Erwin Bolthausen. "An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model". **in** *Communications in Mathematical Physics*: 325.1 (2014), **pages** 333–366.

[3] Sourav Chatterjee. "A simple invariance theorem". **in** *arXiv preprint math/0508213*: (2005).

[4] Oliver Y Feng **andothers**. "A unifying tutorial on approximate message passing". **in** *Foundations and Trends® in Machine Learning*: 15.4 (2022), **pages** 335–536.

[5] Charles W Fox **and** Stephen J Roberts. "A tutorial on variational Bayesian inference". **in** *Artificial intelligence review*: 38 (2012), **pages** 85–95.

[6] Francesco Guerra **and** Fabio Lucio Toninelli. "The thermodynamic limit in mean field spin glass models". **in** *Communications in Mathematical Physics*: 230 (2002), **pages** 71–79.

[7] Andrea Montanari **and** Subhabrata Sen. "A short tutorial on mean-field spin glass techniques for non-physicists". **in** *arXiv preprint arXiv:2204.02909*: (2022).

[8] Dmitry Panchenko. *The sherrington-kirkpatrick model*. Springer Science & Business Media, 2013.

[9] Michel Talagrand. "The generalized Parisi formula". **in** *Comptes Rendus Mathematique*: 337.2 (2003), **pages** 111–114.