Stat 217, Spring 2023 *Jiaze Qiu* (jiazeqiu@g.harvard.edu)

## Section 7: One-step replica symmetry breaking (1RSB)

- Sections: Wed, 7:30-8:30pm (SC 705); OHs: Wed 8:30-9:30pm (SC 316.07).
- All the section materials (handouts & solutions) can be found either on Canvas or here.

In 1RSB, we divide the indices $[r] = \{1, \cdots, r\}$ into $k := r/m$ blocks, each block having $m$ elements. We fix a partition $[r] = \cup_{\ell=1}^{r/m} \mathcal{I}_\ell, |\mathcal{I}_\ell| = m$ and set

$$\begin{aligned} Q_{aa} &= 1 \quad a \in \{0, 1, \cdots, r/m\} \\ Q_{0a} &= b \quad a \in \{1, \cdots, r/m\} \\ Q_{ab} = q_1 &\quad a, b \in \mathcal{I}_\ell, \ell \in \{1, \cdots, r/m\} \\ Q_{ab} &= q_0 \quad \text{otherwise.} \end{aligned}$$

In addition, we partition the matrix and denote by $\boldsymbol{Q}$ as the $r \times r$ submatrix corresponding to the indices $\{1, \cdots, r\}$.

1. Prove the following equality,

$$\operatorname{Tr}\log\left(\boldsymbol{Q}^{\text{1RSB}}\right) = \log\left(1 - b^2 \left\langle \mathbf{1}, \tilde{\boldsymbol{Q}}^{-1}\mathbf{1}\right\rangle\right) + \operatorname{Tr}\log(\tilde{\boldsymbol{Q}}) \tag{1}$$

   Hint: use the Schur-complement formula.

   **Lemma 1.** *Assume both $A$ and $D$ are square matrices and $D^{-1}$ exists. Then*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_p & BD^{-1} \\ 0 & I_q \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I_p & 0 \\ D^{-1}C & I_q \end{bmatrix}$$

2. Show that

$$\begin{aligned} \operatorname{Tr}\log\left(\boldsymbol{Q}^{\text{1RSB}}\right) = &\log\left(1 - \frac{b^2 r}{1 + (m-1)q_1 + (r-m)q_0}\right) + \log\left(1 + (m-1)q_1 + (r-m)q_0\right) \\ &+ \left(\frac{r}{m} - 1\right)\log\left(1 - q_1 + m\left(q_1 - q_0\right)\right) + \frac{r}{m}(m-1)\log\left(1 - q_1\right). \end{aligned} \tag{2}$$

3. Recall that

$$S(\boldsymbol{Q}) \equiv h\sum_{a=1}^{r} Q_{0,a} + \frac{\beta\lambda}{\sqrt{2(k!)}}\sum_{a=1}^{r} Q_{0a}^k + \frac{\beta^2}{4}\sum_{a,b=1}^{r} Q_{ab}^k + \frac{1}{2}\operatorname{Tr}\log(\boldsymbol{Q}) \tag{3}$$

and

$$\Psi_{1\mathrm{RSB}}\left(b, q_0, q_1, m\right) = \lim_{r\to 0}\frac{1}{r}S\left(Q^{1\mathrm{RSB}}\right). \tag{4}$$

Based on these definitions, show that

$$\begin{aligned}\Psi_{1\mathrm{RSB}}\left(b, q_0, q_1, m\right) = {} & \frac{\beta\lambda}{\sqrt{2(k!)}}b^k + \frac{\beta^2}{4}\left[1-(1-m)q_1^k - mq_0^k\right] + \frac{1}{2}\frac{q_0-b^2}{1-(1-m)q_1-mq_0} \\ & + \frac{1}{2m}\log\left(1-(1-m)q_1-mq_0\right) - \frac{1-m}{2m}\log\left(1-q_1\right).\end{aligned} \tag{5}$$

# References


[1] Mohsen Bayati **and** Andrea Montanari. “The dynamics of message passing on dense graphs, with applications to compressed sensing”. **in***IEEE Transactions on Information Theory*: 57.2 (2011), **pages** 764–785.

[2] Erwin Bolthausen. “An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model”. **in***Communications in Mathematical Physics*: 325.1 (2014), **pages** 333–366.

[3] Oliver Y Feng **andothers**. “A unifying tutorial on approximate message passing”. **in***Foundations and Trends® in Machine Learning*: 15.4 (2022), **pages** 335–536.

[4] Charles W Fox **and** Stephen J Roberts. “A tutorial on variational Bayesian inference”. **in***Artificial intelligence review*: 38 (2012), **pages** 85–95.

[5] Andrea Montanari **and** Subhabrata Sen. “A short tutorial on mean-field spin glass techniques for non-physicists”. **in***arXiv preprint arXiv:2204.02909*: (2022).

[6] Dmitry Panchenko. *The sherrington-kirkpatrick model*. Springer Science & Business Media, 2013.