Stat 217, Spring 2023 *Jiaze Qiu* (jiazeqiu@g.harvard.edu)

Section 6: Conditioning and Bolthausen's Lemma

- Sections: Wed, 7:30-8:30pm (SC 705); OHs: Wed 8:30-9:30pm (SC 316.07).
- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Bolthausen's Lemma

Lemma 1. *Let $\tilde{Z}$ be an $n \times n$ symmetric real matrix with independent entries, such that $\tilde{Z}_{ij} \sim \mathcal{N}(0,1)$ and $\tilde{Z}_{ii} \sim \mathcal{N}(0,2)$. Let $g \sim \mathcal{N}(0, I_n) \in \mathbb{R}^{n \times n}, \gamma \sim N(0,2)$ and $b \in \mathbb{R}^n$ with $\|b\|_2 = 1$. Assume they are all independent. Then*

$$Z := (I - P_b)gb^T + bg^T(I - P_b) + \gamma bb^T + (I - P_b)\tilde{Z}(I - P_b)$$

is again a symmetric Gaussian matrix and it has the same distribution as $\tilde{Z}$.

1. Prove the lemma above. Hint: WLOG, we can take $b = e_1$.

2. What is the intuition behind this lemma?

2 An iterative construction of AMP

The Bolthausen's algorithm can be written as

$$x^{(t+1)} = f_t\left(\frac{Z}{\sqrt{n}}x^{(t)} - \alpha_t x^{(t-1)}\right)$$

with $x^{(0)} = \vec{0}$ and $x^{(1)} = \vec{1}$.

1. For $t = 2$ and $t = 3$, try to characterize the approximate distribution of $x^{(t)}$ using the lemma above. In order to cancel out the non-Gaussian part, what α_t should we choose?

References

[1] Mohsen Bayati **and** Andrea Montanari. "The dynamics of message passing on dense graphs, with applications to compressed sensing". **in***IEEE Transactions on Information Theory*: 57.2 (2011), **pages** 764–785.

[2] Erwin Bolthausen. "An iterative construction of solutions of the TAP equations for the Sherrington–Kirkpatrick model". **in***Communications in Mathematical Physics*: 325.1 (2014), **pages** 333–366.

[3] Oliver Y Feng **andothers**. "A unifying tutorial on approximate message passing". **in***Foundations and Trends® in Machine Learning*: 15.4 (2022), **pages** 335–536.

[4] Charles W Fox **and** Stephen J Roberts. "A tutorial on variational Bayesian inference". **in***Artificial intelligence review*: 38 (2012), **pages** 85–95.

[5] Andrea Montanari **and** Subhabrata Sen. "A short tutorial on mean-field spin glass techniques for non-physicists". **in***arXiv preprint arXiv:2204.02909*: (2022).

[6] Dmitry Panchenko. *The sherrington-kirkpatrick model*. Springer Science & Business Media, 2013.