Section 9 (Stat 171)

- TF: Jiaze Qiu
  - Email: jiazeqiug.harvard.edu
  - Section/OH: Wed 9-12 pm (ET)
- TF: Fernando Vicente
  - Email: fernando_vicente@fas.harvard.edu
  - Section/OH: Mon 7-9 pm and Thu 8-10 pm (ET)

- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

- A stochastic process $(Y_t)_{t\geq 0}$ is a **martingale**, if for all $t \geq 0$
  1. $E[Y_t \mid Y_r, 0 \leq r \leq s] = Y_s, 0 \leq s \leq t$
  2. $E[|Y_t|] < \infty$
- Specifically, for a discrete-time martingale $Y_0, Y_1, \ldots,$
  1. $E[Y_{n+1} \mid Y_0, \ldots, Y_n] = Y_n, n \geq 0$
  2. $E[|Y_n|] < \infty$
- Note that this deals with expectations, while the Markovian property deals with probability.
- We can also define martingale wrt another process: Let $(Y_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ be two stochastic processes. We say $(Y_n)_{n\geq 0}$ is a martingale wrt $(X_n)_{n\geq 0}$ if for all $n \geq 0$,
  1. $E[|Y_n|] < \infty$
  2. $E[Y_{n+1} \mid X_0, \ldots, X_n] = Y_n$

2 Preview: Stopping time

- For a stochastic process $(Y_t)_{t\geq 0}$, a nonnegative random variable T is a stopping time if for each t, the event $\{T \leq t\}$ can be determined from $\{Y_s, 0 \leq s \leq t\}$. That is, if the outcomes of Y_s are known for $0 \leq s \leq t$, then it can be determined whether or not$\{T \leq t\}$ occurs.
- Optional Stopping Theorem: Let $(Y_t)_{t\geq 0}$ be a martingale. Assume that T is a stopping time. Then, $E(Y_T) = E(Y_0)$ if one of the following is satisfied. 1. T is bounded. That is, $T \leq c$, for some constant c. 2. $P(T < \infty) = 1$ and $E(|Y_t|) \leq c$, for some constant c, whenever $T > t$

3 Practice problems

3.1 Asymmetric Random Walk

1. (a) Consider a random walk on the integers. At each step, it moves to the right with probability $p \in (0,1), p \neq 1/2$, and moves to the left with probability $q = 1-p$. Let the process start from 0 , and let S_n be its position at the n-th step. Show that $M_n = (q/p)^{S_n}$ is a martingale.

Solutions.

Since $|S_n| \leqslant n$,

$$M_n = \left(\frac{q}{p}\right)^{S_n} \leqslant \left(\frac{\max(p,q)}{\min(p,q)}\right)^n$$
$$\Rightarrow E\left|M_n\right| \leqslant \left(\frac{\max(p,q)}{\min(p,q)}\right)^n < \infty$$

Let $X_n = S_n - S_{n-1} = \begin{cases} 1 & \text{w.p.} \quad p \\ -1 & \text{w.p.} \quad q \end{cases}$

$$\Rightarrow X_n \sim 2\,\mathrm{Bern}(p) - 1$$

Thus,

$$\begin{aligned}
E\left[M_{n+1} \mid M_0, \ldots, M_n\right] &= E\left[\left(\frac{q}{p}\right)^{S_{n+1}} \mid M_{0,\ldots,} M_n\right] \\
&= E\left[\left(\frac{q}{P}\right)^{S_n} \cdot \left(\frac{q}{p}\right)^{X_{n+1}} \mid M_0, \ldots, M_n\right] \\
&= E\left[M_n \left(\frac{q}{p}\right)^{X_{n+1}} \mid M_0, \ldots, M_n\right] \\
&= M_n E\left[\left(\frac{q}{p}\right)^{X_{n+1}} \mid M_0, \ldots, M_n\right] \\
&= M_n E\left[\left(\frac{q}{p}\right)^{X_{n+1}}\right] \\
&= M_n \cdot \left(\left(\frac{q}{p}\right) \cdot p + \left(\frac{q}{p}\right)^{-1} \cdot q\right) \\
&= M_n \cdot (q+p) \\
&= M_n
\end{aligned}$$

(b) Let $Y_1, \cdots, Y_n, \cdots$ be independent random variables. Let

$$Z_n = \prod_{k=1}^n \frac{e^{sY_k}}{Ee^{sY_k}} \tag{1}$$

Show that for any constant s, $(Z_n)_{n \geq 0}$ is a martingale. Could you provide some intuitive explanations for this result? How does it connect to part (a)?

Solutions.

$$\begin{aligned}
E\left[Z_{n+1} \mid Z_0, \ldots, Z_n\right] =& E\left[\prod_{k=1}^{n+1} \frac{e^{sY_k}}{Ee^{sY_k}} \mid Z_0, \ldots, Z_n\right] \\
=& \frac{\prod_{k=1}^n e^{sY_k}}{\prod_{k=1} Ee^{sY_k}} E\left[e^{sY_{n+1}} \mid Z_0, \ldots, Z_n\right] \\
=& Z_n \frac{1}{E\left(e^{sY_{n+1}}\right)} E\left(e^{sY_{n+1}}\right) = Z_n
\end{aligned}$$

3.2 Constructing Martingales

2. Let $(Y_n)_{n\geq 0}$ be a stochastic process with finite absolute expectations. Define

$$Z_n = Y_n - \sum_{k=0}^{n-1} \left(E\left[Y_{k+1} \mid Y_0, \dots, Y_k\right] - Y_k\right)$$

Show that $(Z_n)_{n\geq 0}$ is a martingale wrt $(Y_n)_{n\geq 0}$.
Solutions.

$$\begin{aligned}
&E\left[Z_{n+1} \mid Y_0, \dots, Y_n\right] \\
=&E\left[Y_{n+1} \mid Y_0, \dots, Y_n\right] - \sum_{k=0}^{n} \left(E\left[E\left[Y_{k+1} \mid Y_0, \dots, Y_k\right] \mid Y_0, \dots, Y_n\right] - E\left[Y_k \mid Y_0, \dots, Y_n\right]\right) \\
=&E\left[Y_{n+1} \mid Y_0, \dots, Y_n\right] - \left(E\left[Y_{n+1} \mid Y_0, \dots, Y_n\right] - Y_n + \sum_{k=0}^{n-1} \left(E\left[Y_{k+1} \mid Y_0, \dots, Y_k\right] - Y_k\right)\right) \qquad (2) \\
=&Y_n - \sum_{k=0}^{n-1} \left(E\left[Y_{k+1} \mid Y_0, \dots, Y_k\right] - Y_k\right) \\
=&Z_n
\end{aligned}$$

3.3 Martingale Betting Strategy

Let $(\eta_n)_{n\geq 1}$ be i.i.d. Bern(1/2) random variables, i.e., $P(\eta_n = 1) = P(\eta_n = -1) = 1/2$. Define

$$\xi_n = \xi_0 + \sum_{k=1}^{n} b_k\left(\xi_0, \eta_1, \cdots, \eta_{k-1}\right) \cdot \eta_k \tag{3}$$

where for any k, b_k is a deterministic function. For simplicity we can assume $\xi_0 = 0$.
(a) Show that

$$E\left[\xi_{n+1} \mid \xi_0, \eta_1, \cdots, \eta_n\right] - \xi_n = 0 \tag{4}$$

(b) Try to come up with a gambling explanation for this mathematical model.
Solutions. η_n represents the outcome of one fair game. b_k as the amount you bet for the k-th game, or equivalently the betting strategy. ξ_n as how much money you have lost or won until time n.
(c) Could you find a 'always winning' strategy for your gambler by choosing your b_k? Please try to formulate everything into rigorous mathematical language.
Solutions. Once win, stop and walk away. If lose, double the bet and continue gambling. Then your gambler will with probability 1 end up wining exactly one dollar.
(d) Is your answer for (c) a contradiction to the 'fairness' intuition of martingale?
Solutions. No, since the fairness intuition of martingale only holds for 'finite time'.