Section 9 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

- A stochastic process $(Y_t)_{t\geq 0}$ is a **martingale**, if for all $t \geq 0$
  1. $E[Y_t \mid Y_r, 0 \leq r \leq s] = Y_s, 0 \leq s \leq t$
  2. $E[|Y_t|] < \infty$
- Specifically, for a discrete-time martingale $Y_0, Y_1, \ldots,$
  1. $E[Y_{n+1} \mid Y_0, \ldots, Y_n] = Y_n, n \geq 0$
  2. $E[|Y_n|] < \infty$
- Note that this deals with expectations, while the Markovian property deals with probability.
- We can also define martingale wrt another process: Let $(Y_n)_{n\geq 0}$ and $(X_n)_{n\geq 0}$ be two stochastic processes. We say $(Y_n)_{n\geq 0}$ is a martingale wrt $(X_n)_{n\geq 0}$ if for all $n \geq 0$,
  1. $E[|Y_n|] < \infty$
  2. $E[Y_{n+1} \mid X_0, \ldots, X_n] = Y_n$

2 Preview: Stopping time

- For a stochastic process $(Y_t)_{t\geq 0}$, a nonnegative random variable T is a stopping time if for each t, the event $\{T \leq t\}$ can be determined from $\{Y_s, 0 \leq s \leq t\}$. That is, if the outcomes of Y_s are known for $0 \leq s \leq t$, then it can be determined whether or not$\{T \leq t\}$ occurs.
- Optional Stopping Theorem: Let $(Y_t)_{t\geq 0}$ be a martingale. Assume that T is a stopping time. Then, $E(Y_T) = E(Y_0)$ if one of the following is satisfied.
  1. T is bounded. That is, $T \leq c$, for some constant c.
  2. $P(T < \infty) = 1$ and $E(|Y_t|) \leq c$, for some constant c, whenever $T > t$

3 Practice problems

3.1 Asymmetric Random Walk

1. (a) Consider a random walk on the integers. At each step, it moves to the right with probability $p \in (0,1), p \neq 1/2$, and moves to the left with probability $q = 1 - p$. Let the process start from 0 , and let S_n be its position at the n-th step. Show that $M_n = (q/p)^{S_n}$ is a martingale.
(b) Let $Y_1, \cdots, Y_n, \cdots$ be independent random variables. Let

$$Z_n = \prod_{k=1}^{n} \frac{e^{sY_k}}{Ee^{sY_k}} \tag{1}$$

Show that for any constant s, $(Z_n)_{n\geq 0}$ is a martingale. Could you provide some intuitive explanations for this result? How does it connect to part (a)?

3.2 Constructing Martingales

2. Let $(Y_n)_{n\geq 0}$ be a stochastic process with finite absolute expectations. Define

$$Z_n = Y_n - \sum_{k=0}^{n-1} \left(E\left[Y_{k+1} \mid Y_0, \dots, Y_k\right] - Y_k\right)$$

Show that $(Z_n)_{n\geq 0}$ is a martingale wrt $(Y_n)_{n\geq 0}$.

3.3 Martingale Betting Strategy

Let $(\eta_n)_{n\geq 1}$ be i.i.d. Bern(1/2) random variables, i.e., $P(\eta_n = 1) = P(\eta_n = -1) = 1/2$. Define

$$\xi_n = \xi_0 + \sum_{k=1}^{n} b_k\left(\xi_0, \eta_1, \cdots, \eta_{k-1}\right) \cdot \eta_k \tag{2}$$

where for any k, b_k is a deterministic function. For simplicity we can assume $\xi_0 = 0$.

(a) Show that

$$E\left[\xi_{n+1} \mid \xi_0, \eta_1, \cdots, \eta_n\right] - \xi_n = 0 \tag{3}$$

(b) Try to come up with a gambling explanation for this mathematical model.

(c) Could you find a 'always winning' strategy for your gambler by choosing your b_k? Please try to formulate everything into rigorous mathematical language.

(d) Is your answer for (c) a contradiction to the 'fairness' intuition of martingale?