Section 7 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Poisson Processes Review

- A **counting process** $(N_t)_{t\geq 0}$ is a stochastic process such that:
  - $N_t \geq 0$ (non-negative)
  - N_t is an integer
  - $s \leq t \to N_s \leq N_t$ (non-decreasing)
- A **Poisson process** with parameter λ (not to be confused with poisson random variable) is a special type of counting process that further satisfies:
  - $N_0 = 0$
  - $N_t \sim \text{Pois}(\lambda t)$
  - For $s, t > 0, N_{t+s} - N_s \sim N_t \sim \text{Pois}(\lambda t)$
- A poisson process can also be characterized using exponential random variables:

  $$N_t = \max\{n : X_1 + X_2 + \dots + X_n \leq t\}, \quad X_1, X_2, \dots \sim \text{Exp}(\lambda)$$

  This implies:

  $$X_k = S_k - S_{k-1}, \quad S_k = X_1 + X_2 + \dots$$

  Where S_k also denotes the kth arrival time in a Poisson process.
- **Thinning/Superposition**: Both the thinned and superpositioned Poisson processes are also Poisson processes.
- **Conditional Poisson Process**: In addition to being related to exponential random variables, Poisson processes are also related to uniform random variables in the form of order statistics. Let $S_1, S_2, \dots S_n$ be the arrival times of a Poisson process:
  - $(S_1, S_2, \dots S_n) \mid N_t = n \sim$ order statistics on $Unif[0, t]$
  - $f_{(S_1,S_2,\dots S_n)|N_t=n}(S_1, S_2, \dots S_n) = \frac{n!}{t^n}$ where $0 \leq S_1 \leq \dots \leq S_n$
- **Spatial Poisson Process**: A spatial Poisson process $(N_A)_{A\subseteq\mathbb{R}^d}$ with parameter λ satisfies the following:
  - For each bounded set $A \subseteq \mathbb{R}^d, N_A \sim \text{Pois}(\lambda|A|)$
  - If A and B are disjoint, N_A is independent from N_B

  Note that a spatial Poisson process is essentially a Poisson process generalized to higher dimensions.

2 Practice problems

2.1 Midterm 1 Problem 4(b)

Suppose a Markov chain $X_0, X_1, ...$ has the transition matrix

$$P = \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Given that the chain starts in state 1 and is eventually absorbed into state 3, what is the expected number of steps until the chain is absorbed.

Solutions. Please refer to the midterm solutions.

2.2 Spatial Poisson

Consider a spatial Poisson process with parameter λ in $\mathbb{R}^3$. Find the distribution of the distance from the origin to the nearest point/event. How is it related to the exponential distribution?

Solutions. Let this distance be D. Let $B_r(y)$ be the ball centered at y with radius r.

$$\begin{aligned} P(D > x) &= P\left(N_{B_x(0)} = 0\right) \\ &= \exp\left(-\lambda \frac{4}{3}\pi x^3\right), x > 0 \end{aligned}$$

Hence, the CDF is

$$P(D \leq x) = 1 - \exp\left(-\lambda \frac{4}{3}\pi x^3\right), x > 0$$

Note that

$$P\left(D^3 \leq x\right) = P\left(D \leq x^{1/3}\right) = 1 - \exp\left(-\lambda \frac{4}{3}\pi x\right), x > 0$$

Thus, $D^3 \sim \text{Exp}\left(\frac{4\pi\lambda}{3}\right)$.

2.3 Conditional Poisson

Let Λ be a positive random variable with PDF p. Let $\left(N_t\right)_{t\geq 0}$ be a counting process, such that given $\Lambda = \lambda, \left(N_t\right)_{t\geq 0}$ is a Poisson process with rate λ.

1. Find the PMF for N_t.

Solutions. Conditional on λ.

$$\begin{aligned} P\left(N_t = n\right) &= \int_0^\infty P\left(N_t = n \mid \Lambda = \lambda\right) p(\lambda)\lambda \\ &= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} p(\lambda)\lambda \end{aligned}$$

2. If for all $s, t > 0, N_{t+s} - N_s \sim N_t$, our process is said to have stationary increments. If for any $0 \leq q < r \leq s < t, N_t - N_s$ is independent of $N_r - N_q$, it's said to have independent increments. Show that the our process has stationary increments, and briefly explain why it doesn't have independent increments in general.
Solution. Again, we condition on Λ.

$$\begin{aligned} P\left(N_{t+s} - N_s = n\right) &= \int_0^\infty P\left(N_{t+s} - N_s = n \mid \Lambda = \lambda\right) p(\lambda)\lambda \\ &= \int_0^\infty P\left(N_t = n \mid \Lambda = \lambda\right) p(\lambda)\lambda \\ &= P\left(N_t = n\right) \end{aligned}$$

This shows stationary increments. It doesn't have independent increments because when you know (condition on) $N_r - N_q$, you have some additional information on Λ (think about Bayes rule). Thus, conditional on that, the distribution of $N_t - N_s$ should be calculated with the updated information of Λ, hence different from its unconditional distribution.