Section 4 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

Consider a population with each individual producing $k \geq 0$ new offspring by the end of its lifetime with probability α_k independently. Let Z_n be the number of individuals of the nth generation (we usually assume $Z_0 = 1$). This Markov chain is called a branching process.

- In a branching process, the zero state is an absorbing state and the countably infinite non-zero states are all transient.
- The process is completely described by the offspring distribution $\alpha = (\alpha_0, \alpha_1, ...)$ and the initial state $Z_0 = 1$. Let $\mu = \sum_{k=0}^{\infty} k\alpha_k$ be the mean of the distribution. We can further classify the branching process accordingly:
  - Subcritical: $\mu < 1$
  - Critical: $\mu = 1$
  - Supercritical: $\mu > 1$
- $E[Z_n] = \mu E[Z_{n-1}] = \mu^n Z_0 = \mu^n$
- Let σ^2 be the variance of the offspring distribution.

$$Var(Z_n) = \begin{cases} n\sigma^2 & \mu = 1 \\ \frac{\sigma^2\mu^{n-1}(\mu^n - 1)}{\mu - 1} \approx c\mu^{2n} & \mu \neq 1 \end{cases}$$

- Probability Generating Functions (PGFs). For a discrete random variable X taking values in $\{0, 1, ...\}$, the PGF of X is:

$$G(s) = E(s^X) = \sum_{k=0}^{\infty} s^k P(X = k)$$

- Let $G(s)$ be the PGF of the offspring distribution α. Then $G_{Z_n}(s) = G_{Z_{n-1}}(G(s)) = G(G_{Z_{n-1}}(s))$
- **Extinction Probability Thm**
  1. Let $G(s)$ be the PGF of the offspring distribution α. The probability of eventual extinction is the smallest positive root of the equation $s = G(s)$.
  2. If $\mu \leq 1$, that is, in the subcritical and critical cases, the extinction probability is equal to 1.

2 Problems

1. Branching process with binomial offspring distribution. In a branching process the number of offspring per individual has a binomial distribution with parameters $(2, p)$. Starting with a single individual, calculate:
(a) the extinction probability.

(b) the probability that the population becomes extinct for the first time in the second generation.

(c) Suppose that, instead of starting with a single individual, the initial population size is a random variable that is Poisson distributed with mean λ. Show that, in this case, the extinction probability is given, for $p > 1/2$, by

$$\exp\{\lambda(1-2p)/p^2\}.$$

2. Branching process with Poisson offspring distribution. Consider a branching process in which the number of offspring per individual has a Poisson distribution with mean $\lambda, \lambda > 1$. Let π_0 denote the probability that, starting with a single individual, the population eventually becomes extinct. Also, let $a, a < 1$, be such that

$$ae^{-a} = \lambda e^{-\lambda}.$$

(a) Show that $a = \lambda\pi_0$.

(b) Show that, conditional on eventual extinction, the branching process has the same transition probabilities as the branching process in which the number of offspring per individual is Poisson with mean a.