Section 3 (STAT 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

1. Review

- States i and j communicate if i is accessible from j and j is accessible from i.
- A Markov chain is irreducible if it has one communication class.
- Recurrent state: if the Markov chain started in j eventually revisits j: 1) If T is the time it takes to return to j, $P(T < \infty) = 1$; 2) However, it is possible for $\mathbb{E}[T] = \infty$. This is known as null recurrence.
- Transient state: if the Markov chain started in j has a positive probability of never returning to j.
- Periodicity: The period of a state i is the greatest common denominator (gcd) of all integers $n > 0$, for which $P_{ii}(n) > 0$
- An Ergodic Markov chain is irreducible, aperiodic, and all states have finite expected return times. Ergodic Markov chains have unique and positive limiting distributions!
- Lazy Chain: $\tilde{P} = \epsilon I + (1 - \epsilon)P$
- Time Reversibility: Let P be the transition matrix of a Markov chain. If x is a probability distribution which satisfies

$$x_i P_{ij} = x_j P_{ji}, \forall i, j$$

  then x is the stationary distribution, and the Markov chain is reversible.

2. Excercises

2.1 Problem 1

Consider a Markov chain given by the following transition matrix,

$$\begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

- Use first-step analysis to find $E\left[T_1 \mid X_0 = 1\right]$, where $T_1 = \min\left\{n > 0 : X_n = 1\right\}$.
- What is the latent assumption(s) for the first-step analysis approach to work?

2.2 Problem 2

In this problem we will investigate how reducibility of Markov chains interact with sta- tionary distributions and limiting distributions.

Problem: Suppose we are given 2 ergodic markov chains with transition matrices P and Q. We want to combine them into a new Markov chain with transition matrix R :

$$\left[\begin{array}{c|c} (1-\epsilon) * \mathbf{P} & \epsilon * \mathbf{I} \\ \hline 0 & \mathbf{Q} \end{array}\right]$$

- Draw this Markov chain
- How many communication classes are there?
- Does this have a limiting distribution? If so, what would be a reasonable guess in terms of the limiting distributions $Q*$ and $P*$ of Q and P, respectively?
- How does this relate to absorption states?
- Supposed I combined more Markov Chains $P1, P2$, and Q as follows:

$$\left[\begin{array}{c|c|c} (1-\epsilon) * \mathbf{P}1 & \epsilon * \mathbf{I} & 0 \\ \hline 0 & (1-\epsilon) * \mathbf{P}2 & \epsilon * \mathbf{I} \\ \hline 0 & 0 & \mathbf{Q} \end{array}\right]$$

  What can you say about the limiting distribution? What if I chained N distributions?
- How does this compare to the Fundamental Theorem of Ergodic Markov Chains?

2.3 Problem 3

M balls are initially distributed among m urns. At each stage one of the balls is selected at random, taken from whichever urn it is in, and placed, at random, in one of the other $m-1$ urns. Consider the Markov chain whose state at any time is the vector $(n_1, \ldots, n_m)$ where n_i denotes the number of balls in urn i. Guess at the limiting probabilities for this Markov chain and then verify your guess and show at the same time that the Markov chain is time reversible.

2.4 Problem 4 (Symmetric Simple Random Walk on $\mathbb{Z}^2$)

Consider a symmetric simple random walk on $\mathbb{Z}^2$, i.e., it moves to one of its four neighbors with equal probability. Show that it's recurrent.