Stat 171: Week 2

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Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

1 Review

Let $\{X_n\}$ be a Markov chain with transition probability $\mathbf{P}$ and initial distribution α.

- Markov Property: $P(X_{n+1} = x_{n+1}|X_0 = x_0, \cdots, X_n = x_n) = P(X_{n+1} = x_{n+1}|X_n = x_n)$, i.e., given the present, the future is independent of the past.
- Stochastic/one-step transition/markov Matrix: $\mathbf{P}$ s.t. (1) $P_{ij} \geq 0 \forall i, j$ and (2) $\sum_j P_{ij} = 1$
- denote n-step transition matrix by $\mathbf{P}^n$, then $\mathbf{P}^n = (\mathbf{P})^n$, where $\mathbf{P}$ is the one-step transition matrix.
- Distribution of X_n is $\alpha\mathbf{P^n}$
- Limiting distribution (if limit exists): $\lim_{n\to\infty} P_{ij}^n = \lambda_j$
- Stationary distribution: $\pi = \pi\mathbf{P}$

2 Exercises

Problem 1. A component of a computer has an active life, measured in discrete units, that is a random variable T where $P(T = k) = a_k$ for $k = 1, 2, \ldots$. Suppose one starts with a fresh component and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n.

1. Let $p_i = P(X_{n+1} = i + 1 | X_n = i)$ and $q_i = P(X_{n+1} = 0 | X_n = i)$ for all n. Write down the stochastic matrix of $\{X_n\}$

Solution.

$$\mathbf{P} = \begin{pmatrix} q_0 & p_0 & 0 & 0 & 0 & \cdots \\ q_1 & 0 & p_1 & 0 & 0 & \cdots \\ q_2 & 0 & 0 & p_2 & 0 & \cdots \\ q_3 & 0 & 0 & 0 & p_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

2. Solve p_i, q_i in terms of a_i.

Solution. $q_i = \frac{a_{i+1}}{a_{i+1}+a_{i+2}+a_{i+3}+\cdots} = 1 - p_i$

Problem 2. Prove that every stochastic process $\{X(t); t = 0, 1, ...\}$ with independent increments is a Markov process. Feel free to assume $X_o = x_o$ is a constant.

Hint: (Definition of Independent Increments) For $0 \leq t_1 < t_2 \leq t_3 < t_4 \leq ... < t_m$, $(X_{t_2} - X_{t_1}), (X_{t_4} - X_{t_3}), \cdots, (X_m - X_{t_{m-1}})$ are collectively independent.

Solution. Let $n > 0$.

X_0 constant gives two helpful results:

(1) $(X_n - X_{n-1}, X_{n-1} - X_{n-2}, \cdots, X_1 - X_0, X_0)$ are collectively independent for any n.

(2) $\{X_i = x_i\}$ is the same event as $\{X_i - X_0 = x_i - x_0\}$.

$$\begin{aligned}
P(X_n = x_n | X_0 = x_0, \ldots, X_{n-1} = x_{n-1}) &= \frac{P(X_n = x_n, X_0 = x_0, \ldots, X_{n-1} = x_{n-1})}{P(X_0 = x_0, \ldots, X_{n-1} = x_{n-1})} \\
&= \frac{P(X_n - X_{n-1} = x_n - x_{n-1}, \ldots, X_1 - X_0 = x_1 - x_0, X_0 = x_0)}{P(X_{n-1} - X_{n-2} = x_{n-1} - x_{n-2}, \ldots, X_1 - X_0 = x_1 - x_0, X_0 = x_0)} \\
&= \frac{P(X_n - X_{n-1} = x_n - x_{n-1})P(X_{n-1} - X_{n-2} = x_{n-1} - x_{n-2}, \ldots, X_0 = x_0)}{P(X_{n-1} - X_{n-2} = x_{n-1} - x_{n-2}, \ldots, X_1 - X_0 = x_1 - x_0, X_0 = x_0)} \quad \textit{by ind. inc.} \\
&= P(X_n - X_{n-1} = x_n - x_{n-1}) \\
&= \frac{P(X_n - X_{n-1} = x_n - x_{n-1})P(X_{n-1} - X_0 = x_{n-1} - x_0)}{P(X_{n-1} - X_0 = x_{n-1} - x_0)} \\
&= \frac{P(X_n - X_{n-1} = x_n - x_{n-1}, X_{n-1} - X_0 = x_{n-1} - x_0)}{P(X_{n-1} - X_0 = x_{n-1} - x_0)} \\
&= \frac{P(X_n - X_{n-1} = x_n - x_{n-1}, X_{n-1} = x_{n-1})}{P(X_{n-1} = x_{n-1})} \\
&= P(X_n = x_n | X_{n-1} = x_{n-1})
\end{aligned}$$

Problem 3.

Consider a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

1. States i and j communicate if i is accessible from j and j is accessible from i. Find all its communication classes.

Solution. If we draw the Markov Chain we will see that we have 2 communication classes (in section scratchwork). Can also find them by observing that the matrix is in a block matrix format, which alludes that certain groups of nodes only transition to each other.

2. Given

$$\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{5}{7} \\ 0 & 0 & \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$

Does the Markov chain have a limiting distribution? Why?

Solution. No, since the rows are not identical

3. Find all of its stationary distributions.

Solution. Let π be a stationary distribution such that $\pi\mathbf{P}\pi$. If we write $\pi = (\pi_1, \pi_2)$ and rewrite $\mathbf{P}$ using a block matrix format, then we have

$$(\pi_1, \pi_2) \begin{pmatrix} Q & O \\ O & R \end{pmatrix} = (\pi_1 Q, \pi_2 R) = \pi = (\pi_1, \pi_2)$$

Thus we can solve $\pi_1 Q = \pi$ such that $\pi = (1/2, 1/2)$. Similarly, $\pi_2 R = \pi_2$ gives $\pi_2 = (2/7, 5/7)$. For π to be a distribution, we need $\sum_{ij} \pi_{ij} = 1$. Thus, we can assign $\pi = (\alpha\pi_1, \beta\pi_2)$, where $\alpha + \beta = 1$ and $\alpha, \beta \geq 0$.