Section 2 (STAT 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

1. Review

Let $\{X_n\}$ be a Markov chain with transition probability $\mathbf{P}$ and initial distribution α

- Markov Property: $P\left(X_{n+1}=x_{n+1} \mid X_0=x_0, \cdots, X_n=x_n\right)=P\left(X_{n+1}=x_{n+1} \mid X_n=x_n\right)$, i.e., given the present, the future is independent of the past.
- Stochastic/One-step Transition/markov Matrix: $\mathbf{P}$ s.t. $(1) P_{ij} \geq 0, \forall i, j$ and $(2) \sum_j P_{ij}=1$
- Denote n-step transition matrix by $\mathbf{P}^n$, then $\mathbf{P}^n=(\mathbf{P})^n$, where $\mathbf{P}$ is the one-step transition matrix.
- Distribution of X_n is $\alpha \mathbf{P}^{\mathbf{n}}$
- Limiting distribution (if limit exists): $\lim_{n \rightarrow \infty} P_{ij}^n=\lambda_j$
- Stationary distribution: $\pi=\pi \mathbf{P}$

2. Excercises

2.1 Problem 1

A component of a computer has an active life, measured in discrete units, that is a random variable T where $P(T=k)=a_k$ for $k=1,2,\cdots$ Suppose one starts with a fresh component and each component is replaced by a new component upon failure. Let X_n be the age of the component in service at time n.

1. Let $p_i = P\left(X_{n+1}=i+1 \mid X_n=i\right)$ and $q_i = P\left(X_{n+1}=0 \mid X_n=i\right)$ for all n. Write down the stochastic matrix of $\{X_n\}$.
2. Solve p_i, q_i in terms of a_i.

2.2 Problem 2

Prove that every stochastic process $\{X(t); t=0,1,\ldots\}$ with independent increments is a Markov process. Feel free to assume $X_o = x_o$ is a constant.

Hint: (Definition of Independent Increments) For $0 \leq t_1 < t_2 < \ldots < t_k < t < s$, $(X_{t_2}-X_{t_1}, X_{t_3}-X_{t_2}, \ldots, X_s - X_t)$ are collectively independent.

2.3 Problem 3

Consider a Markov chain with transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.4 & 0.6 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0.2 & 0.8 \end{pmatrix}$$

1. States i and j communicate if i is accessible from j and j is accessible from i. Find all its communication classes.
2. Given
$$\lim_{n\to\infty} \mathbf{P}^n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{2}{7} & \frac{5}{7} \\ 0 & 0 & \frac{2}{7} & \frac{5}{7} \end{pmatrix}$$
Does the Markov chain have a limiting distribution? Why?
3. Find all of its stationary distributions.