Section 11 (Stat 171)

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- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review - Ito integral

- We can use Riemann sums to study integrated brownian motion:

$$\int_0^t B_s ds = ?$$

- Similarly, we can study integration with respect to brownian motion:

$$\int_0^t g(s) dB_s = ?$$

- Finally, building up to Ito integration, now consider if we tried to integrate some stochastic process with respect to brownian motion (both integrand and integrator are stochastic processes)

$$\int_0^t X_s dB_s = ?$$

- Formally, the Ito integral is defined as

$$\int_0^t X_s dB_s = \lim_{n\to\infty} \sum_{k=1}^n X_{t_{k-1}} \left(B_{t_k} - B_{t_{k-1}}\right)$$

  - Requirements
    * $\int_0^t E\left(X_s^2\right) ds < \infty$
    * X_t does not depend on $\{B_s : s > t\}$ but can depend on previous s $\{B_s : s \leq t\}$ (X_t is adapted to brownian motion)
  - Properties
    1. For processes $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$, and constants α, β,
    $$\int_0^t \left(\alpha X_s + \beta Y_s\right) dB_s = \alpha \int_0^t X_s dB_s + \beta \int_0^t Y_s dB_s.$$
    2. For $0 < r < t$,
    $$\int_0^t X_s dB_s = \int_0^r X_s dB_s + \int_r^t X_s dB_s.$$
    3. $E\left(I_t\right) = 0$.
    4. $$E\left(I_t^2\right) = \int_0^t E\left(X_s^2\right) ds$$
    5. $(I_t)_{t\geq 0}$ is a martingale with respect to Brownian motion.

2 Practice problems

1. Find the distribution of $\int_0^t e^s dB_s$.

2. Let $f : \mathbb{R} \to \mathbb{R}, T > 0$ be such that $\int_0^T f^2(s)ds < \infty$. Define

$$M_t = \int_0^t f(s)dB_s, \quad 0 \leq t \leq T$$

Prove that $\left\{M_t^2 - \int_0^t f(s)^2 ds : 0 \leq t \leq T\right\}$ is a martingale, with respect to $\{B_\tau : 0 \leq \tau \leq T\}$.

3. Let $\{B_t^1 : t \geq 0\}$ and $\{B_t^2 : t \geq 0\}$ be two independent Brownian motions. Fix $a > 0$, and let $\tau = \inf\{t > 0 : B_t^1 = a\}$. Find $P(B_\tau^2 < a)$.