Section 12 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

Let $\{B_t : t \geq 0\}$ be standard Brownian motion.

- Martingales for continuous-time stochastic processes
  - $\{X_t : t \geq 0\}$ is a martingale if $E[X_{t+h}|(X_s)_{s=0}^t] = X_t$.
  - B_t is a martingale, so we can use the Optional Stopping Theorem for appropriate stopping times.
- Brownian Motion with drift: For $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$, the process $X_t = \mu t + \sigma B_t$ is a Brownian motion with drift μ and variance σ^2. Note that
  - $X_t \sim N(\mu t, \sigma^2 t)$
  - $(X_t - X_s) \sim N(\mu t, \sigma^2 t)$
- General Brownian Bridge: We want the Brownian motion to hit points $(0, x)$ and $(1, y)$. The process $X_t = x + B_t - t(B_1 - (y - x))$ is a Brownian bridge with start point x and end point y over interval $[0, 1]$.
- Geometric Brownian Motion: For $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$, let X_t be a Brownian motion with drift μ and variance σ^2. Then $G_t = G_0 e^{X_T} = G_0 e^{\mu t + \sigma B_t}$ is a Geometric Brownian motion.
  - This can be viewed as a noisy exponential function.
  - Expectation: $E[G_t] = G_0 e^{t(\mu + \frac{\sigma^2}{2})}$
  - Variance: $Var(G_t) = G_0^2 e^{2t(\mu + \frac{\sigma^2}{2})}(e^{t\sigma^2} - 1)$
  - The increment ratios $\log(\frac{G_{t+s}}{G_t}) = (X_{t+s} - X_s)$ are stationary and independent increments.
  - Let $r = \mu + \sigma^2/2$, then the process $\{e^{-rt}G_t : t \geq 0\}$ is a martingale with respect to B_t.
  - We can use Geometric Brownian motion to model stock prices. Notice that the higher the σ the greater the fluctuations. Thus σ is called the volatility and high σ corresponds with high risk.
- Financial Options and Black-Scholes
  - Let $\{Y_t : t \geq 0\}$ be a geometric Brownian motion with drift μ and volatility σ^2, and let $r = \mu + \sigma^2/2$.
  - An option gives the buyer the right to buy shares of a stock at a fixed strike price sometime in the future. We are interested in modeling this with a Geometric Brownian
  - Let $G_0 =$ current price, $K =$strike price, $t =$ expiration date (time until option is exercised), $Y_t =$ value of stock price at time t.
  - Expected profit: $E[\max(G_t - K, 0)] = G_0 e^{t(\mu + \sigma^2/2} P\left(Z < \frac{\beta - \sigma t}{\sqrt{t}}\right) - KP\left(Z > \frac{\beta}{\sqrt{t}}\right)$, where $\beta = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - \mu t)$

- $P = e^{-rt}F$ where F is the future value at t and P is the present day value. Think of our objective as trying to use our future value (which we have a reasonable model for), to calculate our present price.
- Recall that for future price we have a Geometric Brownian motion model. Let Y_t be the future price of a stock t years from today. Using the compounding interest formula:

$$Y_p = e^{-rt}Y_t.$$

  Y_p is a martingale.
- present value: $e^{-rt}\max(Y_t - K, 0)$ since $\max(Y_t - K, 0)$ is the future value of the option.
- $E[e^{-rt}\max(Y_t - K, 0)] = G_0 P\left(Z > \frac{\alpha - \sigma t}{\sqrt{t}}\right) - e^{-rt}KP\left(Z > \frac{\alpha}{\sqrt{t}}\right)$, where $\alpha = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - \mu t) = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - (r - \sigma^2/2)t)$.

2 Problems

1. Running maximum of Brownian motion Find the mean and variance of the maximum value of standard Brownian motion on $[0, t]$.

Solution. Let $M_t = \max_{0 \geq s \geq t} B_s$. From lecture, we know that $M_t \sim \sqrt{t}|Z|$, where $Z \sim N(0,1)$.

$$E[M_t] = E[\sqrt{t}|Z|] = \sqrt{t}E|Z| = 2\sqrt{t}\int_0^\infty \frac{x}{\sqrt{2\pi}}e^{-x^2/2}dx = \sqrt{\frac{2t}{\pi}}\int_0^\infty -\frac{d}{dx}(e^{-x^2/2})dx = \sqrt{\frac{2t}{\pi}}(1) = \sqrt{\frac{2t}{\pi}}$$

$$E(M_t^2) = tEZ^2 = t$$

Thus, $Var(M_t) = E(M_t^2) - (E[M_t])^2 = t - \frac{2t}{\pi}$.

2. Reflection principle Use the reflection principle to show $P(M_t \geq a, B_t \leq a-b) = P(B_t \geq a+b)$ for $a, b > 0$.
Solution. For a Brownian motion path from the origin to $(t, a+b)$, let T_a be the first time the path reaches level a. Create a new path by reflecting the piece on the interval (T_a, t) about the line $y = a$. This creates a Brownian motion path from the origin to $(t, a-b)$, whose maximum value is at least a. The correspondence is invertible, and shows that

$$\{M_t \geq a, B_t \geq a-b\} = \{T_a \leq t, B_t \geq a-b\} = \{T_a \leq t, B_t \geq a+b\} = \{B_t \geq a+b\}$$

from which the results follows.

3. Zeros of Brownian motion Let $0 < r < s < t$.
(a) Assume that standard Brownian motion is not zero in (r, s). Find the probability that standard Brownian motion is not zero in (r, t).
(b) Assume that standard Brownian motion is not zero in $(0, s)$. Find the probability that standard Brownian motion is not zero in $(0, t)$.

Solution.

(a) Denote $A_{r,t}$ the event that there is no zeros between (r, t).

$$\begin{aligned} P(A_{r,t}|A_{r,s}) &= \frac{P(A_{r,t})}{P(A_{r,s})} \\ &= \frac{1 - P(\textit{at least 1 zero in } (r,t))}{1 - P(\textit{at least 1 zero in } (r,s))} \\ &= \frac{1 - \frac{2}{\pi} \arccos \sqrt{r/t}}{1 - \frac{2}{\pi} \arccos \sqrt{r/s}} \\ &= \frac{\arcsin \sqrt{r/t}}{\arcsin \sqrt{r/s}} \end{aligned}$$

(b) Take the limit, as $r \to \infty$, in the result of (a). By l'Hospital's rule, the desired probability is

$$\lim_{r\to\infty} \frac{\arcsin \sqrt{r/t}}{\arcsin \sqrt{r/t}} = \lim_{r\to\infty} \frac{2\sqrt{1-r/s}\sqrt{rs}}{2\sqrt{1-r/t}\sqrt{rt}} = \sqrt{s/t} \lim_{r\to\infty} \sqrt{\frac{1-r/s}{1-r/t}} = \sqrt{s/t}$$

4. Bronian motion absorbed at a From standard Borwnian motion, B_t, let X_t be the process defined by

$$X_t = \begin{cases} B_t, & \text{if } t < T_a \\ a, & \text{if } t \geq T_a \end{cases}$$

where T_a is the first hitting time of $a > 0$. The process $(X_t)_{t \geq 0}$ is called a Brownian motion absorbed at a. The distribution of X_t has a discrete and continuous parts.

a) Show

$$P(X_t = a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

b) For $x < a$, show

$$P(X_t \leq x) = P(B_t \leq x) - P(B_t \leq x - 2a) = \frac{1}{\sqrt{2\pi t}} \int_{x-2a}^{x} e^{-z^2/2t} dz$$

Solution.

(a)

$$P(X_t = a) = P(T_a \leq t) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

(b) Using the reflection principal result from problem 2 gives

$$\begin{aligned} P(X_t \leq x) &= P(M_t \leq a, X_t \leq x) \\ &= P(M_t \leq a, B_t \leq x) \\ &= P(B_t \leq x) - P(M_t \geq a, B_t \leq x) \\ &= P(B_t \leq x) - P(M_t \geq a, B_t \leq a - (a - x)) \\ &= P(B_t \leq x) - P(B_t \geq a + (a - x)) \text{ from problem 2} \\ &= P(B_t \leq x) - P(B_t \leq x - 2a) \text{ by symmetry of normal since } 2a - x > 0 \end{aligned}$$

and the RHS follows.