# Section 12 (Stat 171)


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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

## 1 Review

Let $\{B_t : t \geq 0\}$ be standard Brownian motion.

- Martingales for continuous-time stochastic processes
  - $\{X_t : t \geq 0\}$ is a martingale if $E[X_{t+h}|(X_s)_{s=0}^t] = X_t$.
  - $B_t$ is a martingale, so we can use the Optional Stopping Theorem for appropriate stopping times.
- Brownian Motion with drift: For $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$, the process $X_t = \mu t + \sigma B_t$ is a Brownian motion with drift $\mu$ and variance $\sigma^2$. Note that
  - $X_t \sim N(\mu t, \sigma^2 t)$
  - $(X_t - X_s) \sim N(\mu t, \sigma^2 t)$
- General Brownian Bridge: We want the Brownian motion to hit points $(0, x)$ and $(1, y)$. The process $X_t = x + B_t - t(B_1 - (y - x))$ is a Brownian bridge with start point $x$ and end point $y$ over interval $[0, 1]$.
- Geometric Brownian Motion: For $\mu \in \mathbb{R}, \sigma \in \mathbb{R}^+$, let $X_t$ be a Brownian motion with drift $\mu$ and variance $\sigma^2$. Then $G_t = G_0 e^{X_T} = G_0 e^{\mu t + \sigma B_t}$ is a Geometric Brownian motion.
  - This can be viewed as a noisy exponential function.
  - Expectation: $E[G_t] = G_0 e^{t(\mu + \frac{\sigma^2}{2})}$
  - Variance: $Var(G_t) = G_0^2 e^{2t(\mu + \frac{\sigma^2}{2})}(e^{t\sigma^2} - 1)$
  - The increment ratios $\log(\frac{G_{t+s}}{G_t}) = (X_{t+s} - X_s)$ are stationary and independent increments.
  - Let $r = \mu + \sigma^2/2$, then the process $\{e^{-rt}G_t : t \geq 0\}$ is a martingale with respect to $B_t$.
  - We can use Geometric Brownian motion to model stock prices. Notice that the higher the $\sigma$ the greater the fluctuations. Thus $\sigma$ is called the volatility and high $\sigma$ corresponds with high risk.
- Financial Options and Black-Scholes
  - Let $\{Y_t : t \geq 0\}$ be a geometric Brownian motion with drift $\mu$ and volatility $\sigma^2$, and let $r = \mu + \sigma^2/2$.
  - An option gives the buyer the right to buy shares of a stock at a fixed strike price sometime in the future. We are interested in modeling this with a Geometric Brownian
  - Let $G_0$ = current price, $K$ =strike price, $t$ = expiration date (time until option is exercised), $Y_t$ = value of stock price at time t.
  - Expected profit: $E[\max(G_t - K, 0)] = G_0 e^{t(\mu+\sigma^2/2} P\left(Z < \frac{\beta - \sigma t}{\sqrt{t}}\right) - KP\left(Z > \frac{\beta}{\sqrt{t}}\right)$, where $\beta = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - \mu t)$

- $P = e^{-rt}F$ where F is the future value at t and P is the present day value. Think of our objective as trying to use our future value (which we have a reasonable model for), to calculate our present price.
- Recall that for future price we have a Geometric Brownian motion model. Let $Y_t$ be the future price of a stock $t$ years from today. Using the compounding interest formula:

$$Y_p = e^{-rt}Y_t.$$

  $Y_p$ is a martingale.
- present value: $e^{-rt}\max(Y_t - K, 0)$ since $\max(Y_t - K, 0)$ is the future value of the option.
- $E[e^{-rt}\max(Y_t - K, 0)] = G_0 P\left(Z > \frac{\alpha - \sigma t}{\sqrt{t}}\right) - e^{-rt}KP\left(Z > \frac{\alpha}{\sqrt{t}}\right)$, where $\alpha = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - \mu t) = \frac{1}{\sigma}(\log(\frac{K}{G_0}) - (r - \sigma^2/2)t)$.

## 2 Problems

**1. Running maximum of Brownian motion** Find the mean and variance of the maximum value of standard Brownian motion on $[0, t]$.

**2. Reflection principle** Use the reflection principle to show $P(M_t \geq a, B_t \leq a - b) = P(B_t \geq a + b)$ for $a, b > 0$.

**3. Zeros of Brownian motion** Let $0 < r < s < t$.

(a) Assume that standard Brownian motion is not zero in $(r, s)$. Find the probability that standard Brownian motion is not zero in $(r, t)$.

(b) Assume that standard Brownian motion is not zero in $(0, s)$. Find the probability that standard Brownian motion is not zero in $(0, t)$.

**4. Bronian motion absorbed at a** From standard Borwnian motion, $B_t$, let $X_t$ be the process defined by

$$X_t = \begin{cases} B_t, & \text{if } t < T_a \\ a, & \text{if } t \geq T_a \end{cases}$$

where $T_a$ is the first hitting time of $a > 0$. The process $(X_t)_{t \geq 0}$ is called a Brownian motion absorbed at a. The distribution of $X_t$ has a discrete and continuous parts.

a) Show

$$P(X_t = a) = \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-x^2/2t} dx$$

b) For $x < a$, show

$$P(X_t \leq x) = P(B_t \leq x) - P(B_t \leq x - 2a) = \frac{1}{\sqrt{2\pi t}} \int_{x-2a}^{x} e^{-z^2/2t} dz$$