Section 11 (Stat 171)

- TF: Jiaze Qiu
  - Email: jiazeqiug.harvard.edu
  - Section/OH: Wed 9-12 pm (ET)
- TF: Fernando Vicente
  - Email: fernando_vicente@fas.harvard.edu
  - Section/OH: Mon 7-9 pm and Thu 8-10 pm (ET)

- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

1.1 Definition of BM

A continuous-time stochastic process $(B_t)_{t\geq 0}$ is a standard Brownian motion if it satisfies the following properties:

1. $B_0 = 0$
2. (Normal distribution) For $t > 0, B_t$ has a normal distribution with mean 0 and variance t
3. (Stationary increments) For $s, t > 0, B_{t+s} - B_s$ has the same distribution as B_t.
4. (Independent increments) If $0 \leq q < r \leq s < t$, then $B_t - B_s$ and $B_r - B_q$ are independent random variables.
5. (Continuous paths) The function $t \mapsto B_t$ is continuous, with probability 1.

1.2 General Markov Process

- A continuous stochastic process $(X_t)_{t\geq 0}$ is a Markov process if it obeys the following:

$$P\left(X_{t+s} \leq j \mid X_s = i, X_u = x_u, 0 \leq u < s\right) = P\left(X_{t+s} \leq j \mid X_s = i\right)$$

- For all $s, t \geq 0$ and $j \in R$. As with continuous Markov Chains, it is time-homogenous if the probability does not depend on s:

$$P\left(X_{t+s} \leq j \mid X_s = i\right) = P\left(X_t \leq j \mid X_0 = i\right)$$

- Instead of having a transition matrix we have a transition kernel $K_t(x, \bullet)$ which is a conditional density of $X_t \mid X_0 = x$:

$$P\left(X_t \in (a, b) \mid X_0 = x\right) = \int_a^b K_t(x, y) dy$$

- The transition kernel satisfies the Chapman-Kolmogorov Equation:

$$K_{s+t}(x, y) = \int_{-\infty}^{\infty} K_s(x, z) K_t(z, y) dz$$

- As with stationary distributions before, here we have a notion of stationary densities. A density f is a stationary density if:

$$f(y) = \int_{-\infty}^{\infty} K_t(x, y) f(x) dx$$

- Note that brownian motion does not have a stationary distribution!

1.3 Stopping time

- Strong Markov property: For a stopping time S, $B_{S+t} - B_S$ is again a standard Brownian motion. Further, $\{B_u : 0 \leq u \leq S\}$ and $\{B_{S+t} - B_S : t \geq 0\}$ are independent.
- Reflection Principle:
  - Let $T_a = \min\{t : B_t = a\}$
  - B_{t+T_a} is a Brownian motion started at a
  - $P(B_t > a \mid T_a < t) = \frac{1}{2} = \frac{P(B_t > a)}{P(T_a < t)}$
  - $T_a \sim \text{Inv-Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$
- Recurrence:
  - $P(T_a < \infty) = 1 \quad \forall \quad a$
  - $E[T_a] = \infty \quad \forall \quad a$
- Running maximum:
  - Let $M_t = \max\{B_s, 0 \leq s \leq t\}$
  - $P(M_t > a) = P(T_a > t) = P(|B_t| > a)$
- Zeros:
  - Brownian motion has infinitely many zeroes (follows from recurrence)
  - Let $z_{r,t}$ denote the probability that $B_s = 0$ somewhere on (r, t)
  - $z_{r,t} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{r}{t}}\right)$

2 Practice Problems

Brownian Motion Problems

1. Suppose B_t is a standard Brownian motion, prove the following processes are all Brownian motions.

   (a) $X_t = cB_{t/c^2}$, $c > 0$.

   A continuous-time stochastic process $(B_t)_{t\geq 0}$ is a standard Brownian motion if it satisfies the following properties:

   i. $B_0 = 0$.

   ii. (*Normal distribution*) For $t > 0$, B_t has a normal distribution with mean 0 and variance t.

   iii. (*Stationary increments*) For $s, t > 0$, $B_{t+s} - B_s$ has the same distribution as B_t.

   iv. (*Independent increments*) If $0 \leq q < r \leq s < t$, then $B_t - B_s$ and $B_r - B_q$ are independent random variables.

   v. (*Continuous paths*) The function $t \mapsto B_t$ is continuous, with probability 1.

   Proof.

   i. $X_0 = 0$.

   ii. $B_{t/c^2} \sim N(0, t/c^2)$, $X_t = cB_{t/c^2} \sim N(0, t)$.

   iii. $X_{t+s} - X_s = c(B_{(t+s)/c^2} - B_{s/c^2}) \sim cB_{t/c^2} = X_t$.

   iv. Obvious.

   v. Obvious.

   (b) $Y_t = B_{t+h} - B_h$ for a fixed $h > 0$.

   Proof. All conditions are obvious.

   (c) $Z_t = tB_{1/t}, t > 0$; $Z_0 = 0$.

   Proof.

   i. $Z_0 = 0$.

   ii. $Z_t \sim tN(0, 1/t) \sim N(0, t)$.

   iii.

$$\begin{aligned} Z_{t+s} - Z_s =& (t+s)B_{1/(t+s)} - sB_{1/s} \\ =& (t+s)B_{1/(t+s)} - s(B_{1/s} - B_{1/(t+s)} + B_{1/(t+s)}) \\ =& tB_{1/(t+s)} - s(B_{1/s} - B_{1/(t+s)}). \end{aligned} \tag{1}$$

   $tB_{1/(t+s)} \sim N(0, t^2/(t+s))$, $s(B_{1/s} - B_{1/(t+s)}) \sim N(0, s^2(1/s - 1/(t+s)))$, and they are independent.

$$Z_{t+s} - Z_s \sim N(0, t^2/(t+s) + s^2(1/s - 1/(t+s))) = N(0, t). \tag{2}$$

iv. Because of the joint normality, we just need to show that $\mathrm{Cov}(Z_t - Z_s, Z_r - Z_q) = 0$ for $0 \leq q < r \leq s < t$.

$$\begin{aligned}
&\mathrm{Cov}(Z_t - Z_s, Z_r - Z_q) \\
=&\,\mathrm{Cov}(tB_{1/t} - sB_{1/s}, rB_{1/r} - qB_{1/q}) \\
=&\,tr\,\mathrm{Cov}(B_{1/t}, B_{1/r}) - tq\,\mathrm{Cov}(B_{1/t}, B_{1/q}) - sr\,\mathrm{Cov}(B_{1/s}, B_{1/r}) + sq\,\mathrm{Cov}(B_{1/s}, B_{1/q}) \\
=&\,tr\frac{1}{t} - tq\frac{1}{t} - sr\frac{1}{s} + sq\frac{1}{s} = 0.
\end{aligned} \tag{3}$$

v. We only need to show Z_t is contiunous at $t = 0$, i.e., $\lim_{t\to 0} tB_{1/t} = 0$. Equivalently, $\lim_{t\to\infty} B_t/t = 0$. We do not present a rigorous proof here, but the main idea is $(Z_{a_n})_{n\geq 1}$ and $(B_{b_n})_{n\geq 1}$ has the same distribution for any positive sequence $\{a_n\}$ with 0 as its limit, so

$$\lim_{n\to\infty} Z_{a_n} = \lim_{n\to\infty} B_{a_n} = \lim_{t\to 0} B_t = 0. \tag{4}$$

2. Let B_t be a standard Brownian motion, and $X_t = |B_t|$. You can think of it as the Brownian motion is reflected everytime it attempts to go through 0. Find the CDF of X_t.

   Solution. We will of course try to write everything in terms of B_t. Hence, for $x \geq 0$,

$$\begin{aligned} P(X_t \leq x) =& P(|B_t| \leq x) \\ =& P(B_t \in [-x, x]) \\ =& P(B_t \leq x) - P(B_t \leq -x) \\ =& P(B_t \leq x) - (1 - P(B_t \leq x)) \text{ (symmetry of normal distribution)} \\ =& 2P(B_t \leq x) - 1 \\ =& \frac{2}{\sqrt{2\pi t}} \int_{-\infty}^{x} e^{-u^2/2t} \, du - 1. \end{aligned} \tag{5}$$

3. Suppose B_t is a standard Brownian motion, show that $Y_t = \exp(cB_t - c^2t/2)$ is a martingale.

Proof. First let's see if $E|Y_t| < \infty$. Because Y_t is positive, $|Y_t| = Y_t$.

$$E(Y_t) = e^{-c^2t/2}E[\exp(cB_t)]. \tag{6}$$

Since $B_t \sim N(0,t)$, $E[\exp(cB_t)]$ is the moment generating function of $N(0,t)$ evaluated at c, which is $\exp(c^2t/2)$. Hence, $E(Y_t) = 1 < \infty$. Now for $0 \leq s \leq t$,

$$\begin{aligned}
&E(Y_t|Y_r, 0 \leq r \leq s)\\
=&E(Y_s \exp(c(B_t - B_s) - c^2t/2 + c^2s/2)|Y_r, 0 \leq r \leq s)\\
=&Y_s \exp(-c^2t/2 + c^2s/2)E(\exp(c(B_t - B_s))|Y_r, 0 \leq r \leq s)\\
=&Y_s \exp(-c^2t/2 + c^2s/2)E(\exp(c(B_t - B_s)))\\
=&Y_s \exp(-c^2t/2 + c^2s/2)\exp(c^2(t-s)/2) = Y_s.
\end{aligned} \tag{7}$$

4. (Brownian motion with a drift) $X_t = B_t + \mu t$ is called a Brownian motion with drift coefficient μ, where B_t is a standard Brownian motion.

   (a) Find the probability that X_t hits A before $-B$, $A, B > 0$. (Hint: use the martingale in the previous problem.)

   Solution. Let $T = \inf\{t \geq 0 : X_t = A \text{ or } -B\}$ be a stopping time. We want to find $p_A = P(X_T = A)$. From the previous problem, we know that $Y_t = \exp(cB_t - c^2 t/2) = \exp(cX_t - c\mu t - c^2 t/2)$ is a martingale. It is easily checked that the second condition of the optional stopping theorem is met. So

$$\begin{aligned} E(Y_0) = 1 =& E(Y_T) \\ =& E(\exp(cX_T - c\mu T - c^2 T/2)). \end{aligned} \tag{8}$$

   By letting $c = -2\mu$, we cancel out the last two terms, and obtain

$$1 = E(\exp(-2\mu X_T)) = p_A \exp(-2\mu A) + (1 - p_A)\exp(2\mu B). \tag{9}$$

   Solving the equation gives

$$p_A = \frac{e^{2\mu B} - 1}{e^{2\mu B} - e^{-2\mu A}}. \tag{10}$$

   (b) Let $T = \inf\{t \geq 0 : X_t = A \text{ or } -B\}$ be the hitting time of $\{A, -B\}$. Find $E(T)$.

   Solution. $B_t = X_t - \mu t$ is known to be a martingale. Again, the second condition of the optional stopping theorem is met. So

$$\begin{aligned} E(B_0) = 0 =& E(B_T) \\ =& E(X_T - \mu T) \\ =& E(X_T) - \mu E(T) \\ =& p_A A + (1 - p_A)(-B) - \mu E(T). \end{aligned} \tag{11}$$

   Plug in p_A, and we get

$$E(T) = \frac{Ae^{2\mu B} + Be^{-2\mu A} - A - B}{\mu(e^{2\mu B} - e^{-2\mu A})}. \tag{12}$$