Section 11 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Review

1.1 Definition of BM

A continuous-time stochastic process $(B_t)_{t\geq 0}$ is a standard Brownian motion if it satisfies the following properties:

1. $B_0 = 0$
2. (Normal distribution) For $t > 0, B_t$ has a normal distribution with mean 0 and variance t
3. (Stationary increments) For $s, t > 0, B_{t+s} - B_s$ has the same distribution as B_t.
4. (Independent increments) If $0 \leq q < r \leq s < t$, then $B_t - B_s$ and $B_r - B_q$ are independent random variables.
5. (Continuous paths) The function $t \mapsto B_t$ is continuous, with probability 1.

1.2 General Markov Process

- A continuous stochastic process $(X_t)_{t\geq 0}$ is a Markov process if it obeys the following:

$$P\left(X_{t+s} \leq j \mid X_s = i, X_u = x_u, 0 \leq u < s\right) = P\left(X_{t+s} \leq j \mid X_s = i\right)$$

- For all $s, t \geq 0$ and $j \in R$. As with continuous Markov Chains, it is time-homogenous if the probability does not depend on s:

$$P\left(X_{t+s} \leq j \mid X_s = i\right) = P\left(X_t \leq j \mid X_0 = i\right)$$

- Instead of having a transition matrix we have a transition kernel $K_t(x, \bullet)$ which is a conditional density of $X_t \mid X_0 = x$:

$$P\left(X_t \in (a, b) \mid X_0 = x\right) = \int_a^b K_t(x, y)dy$$

- The transition kernel satisfies the Chapman-Kolmogorov Equation:

$$K_{s+t}(x, y) = \int_{-\infty}^{\infty} K_s(x, z)K_t(z, y)dz$$

- As with stationary distributions before, here we have a notion of stationary densities. A density f is a stationary density if:

$$f(y) = \int_{-\infty}^{\infty} K_t(x, y)f(x)dx$$

- Note that brownian motion does not have a stationary distribution!

1.3 Stopping time

- Strong Markov property: For a stopping time S, $B_{S+t} - B_S$ is again a standard Brownian motion. Further, $\{B_u : 0 \leq u \leq S\}$ and $\{B_{S+t} - B_S : t \geq 0\}$ are independent.
- Reflection Principle:
  - Let $T_a = \min\{t : B_t = a\}$
  - B_{t+T_a} is a Brownian motion started at a
  - $P(B_t > a \mid T_a < t) = \frac{1}{2} = \frac{P(B_t > a)}{P(T_a < t)}$
  - $T_a \sim \text{Inv-Gamma}\left(\frac{1}{2}, \frac{a^2}{2}\right)$
- Recurrence:
  - $P(T_a < \infty) = 1 \quad \forall \quad a$
  - $E[T_a] = \infty \quad \forall \quad a$
- Running maximum:
  - Let $M_t = \max\{B_s, 0 \leq s \leq t\}$
  - $P(M_t > a) = P(T_a > t) = P(|B_t| > a)$
- Zeros:
  - Brownian motion has infinitely many zeroes (follows from recurrence)
  - Let $z_{r,t}$ denote the probability that $B_s = 0$ somewhere on (r, t)
  - $z_{r,t} = \frac{2}{\pi} \arccos\left(\sqrt{\frac{r}{t}}\right)$

2 Practice problems

2.1 Transformations of BM

1. Suppose B_t is a standard Brownian motion, prove the following processes are all standard Brownian motions.
(a) $X_t = cB_{t/c^2}, c > 0$
(b) $Y_t = B_{t+h} - B_h$ for a fixed $h > 0$.
(c) $Z_t = tB_{1/t}, t > 0; Z_0 = 0$.

2.2

2. Let B_t be a standard Brownian motion, and $X_t = |B_t|$. You can think of it as the Brownian motion is reflected everytime it attempts to go through 0 . Find the CDF of X_t.

2.3 BM and Martingale

3. Suppose B_t is a standard Brownian motion, show that $Y_t = \exp\left(cB_t - c^2 t/2\right)$ is a martingale.

2.4 BM with a drift

4. $X_t = B_t + \mu t$ is called a Brownian motion with drift coefficient μ, where B_t is a standard Brownian motion.
(a) Find the probability that X_t hits A before $-B$, where $A, B > 0$. (Hint: use the martingale in the previous problem.)
(b) Let $T = \inf\{t \geq 0 : X_t = A \text{ or } -B\}$ be the hitting time of $\{A, -B\}$. Find $\mathbf{E}(T)$.