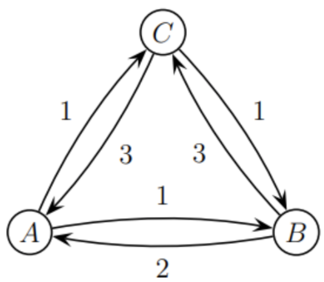
– **Diagonalization**
Suppose Q is diagonalizable, and $Q = SDS^{-1}$, then

$$e^{tQ} = Se^{tD}S^{-1},$$

where

$$e^{tD} = \begin{pmatrix} e^{t\lambda_1} & 0 & \dots & 0 \\ 0 & e^{t\lambda 2} & \dots & 0 \\ \vdots & \vdots & \dots & 0 \\ 0 & 0 & \dots & e^{t\lambda_k} \end{pmatrix}$$

Example 1.1 For the Markov chain with transition rate graph shown below



1. Find the generator matrix.

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 2 & -5 & 3 \\ 3 & 1 & -4 \end{pmatrix}$$

2. Fine the transition matrix of the embedded chain.

$$\tilde{P} = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 2/5 & 0 & 3/5 \\ 3/4 & 1/4 & 0 \end{pmatrix}$$

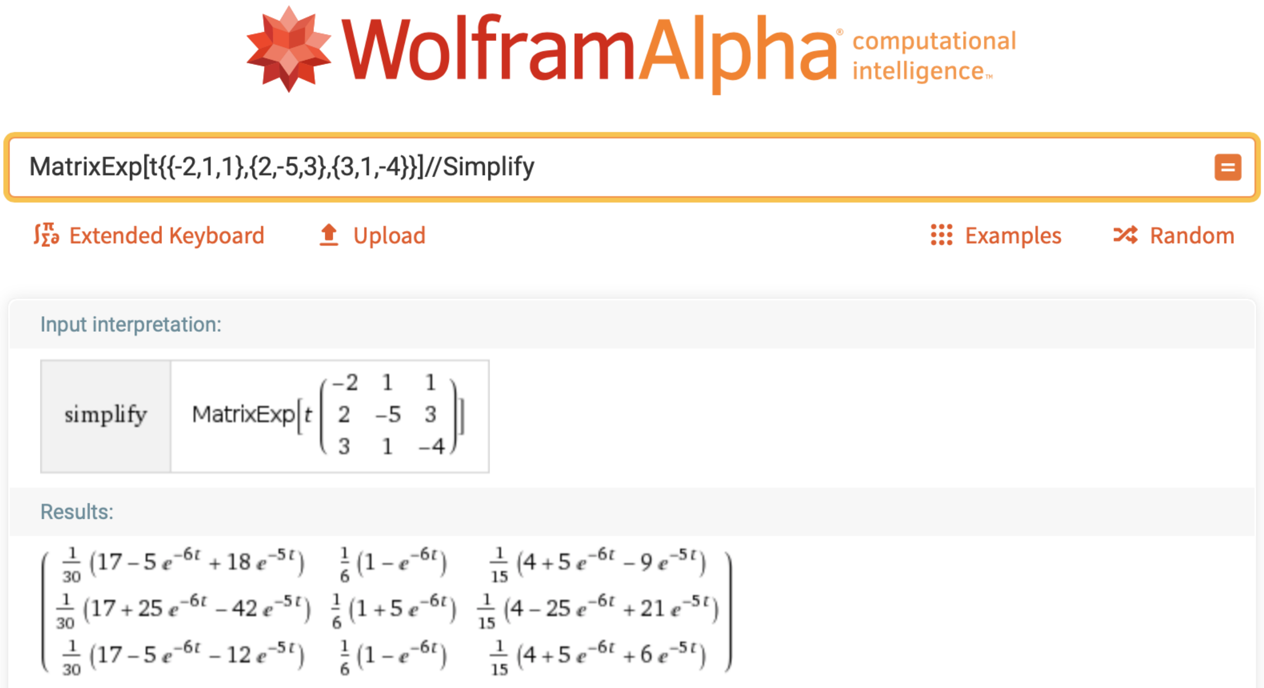
3. Find the transition function.
The generator matrix is diagonalizable with respective matrices of eigenvalues and eigenvectors:

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -6 \end{pmatrix}, S = \begin{pmatrix} 1 & -3 & 1 \\ 1 & 7 & -5 \\ 1 & 2 & 1 \end{pmatrix}$$

Then

$$\begin{aligned} P(t) = e^{tQ} &= Se^{tD}S^{-1} \\ &= \begin{pmatrix} 1 & -3 & 1 \\ 1 & 7 & -5 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-5t} & 0 \\ 0 & 0 & e^{-6t} \end{pmatrix} \frac{1}{30} \begin{pmatrix} 17 & 5 & 8 \\ -6 & 0 & 6 \\ -5 & -5 & 10 \end{pmatrix} \\ &= \frac{1}{30} \begin{pmatrix} 17 - 5e^{-6t} + 18e^{-5t} & 5(1 - e^{-6t}) & 2(4 + 5e^{-6t} - 9e^{-5t}) \\ 17 + 25e^{-6t} - 42e^{-5t} & 5(1 + 5e^{-6t}) & 2(4 - 25e^{-6t} + 21e^{-5t}) \\ 17 - 5e^{-6t} - 12e^{-5t} & 5(1 - e^{-6t}) & 2(4 + 5e^{-6t} + 6e^{-5t}) \end{pmatrix} \end{aligned}$$

You can also compute the matrix exponential with WolframAlpha.



2 Long-term behavior

- **Limiting distribution**
  A probability distribution π is the limiting distribution of a continuous-time Markov chain if for all states i and j,
  $$\lim_{t\to\infty} P_{ij}(t) = \pi_j.$$

- **Stationary distribution**
  A probability distribution π is a stationary distribution if
  $$\pi = \pi P(t), \text{for } t \geq 0.$$

- As in the discrete case, the limiting distribution of a continuous-time Markov chain is a stationary distribution, but the converse is not necessarily true.

- The notions of accessibility, communication, and irreducibility are defined as in the discrete case, with one difference: for a continuous-time Markov chain, **all** states are **aperiodic**.

- **Fundamental Limit Theorem**
  Let $(X_t)_{t\geq 0}$ be a finite, irreducible, continuous-time Markov chain with transition function $P(t)$. Then, there exists a unique stationary distribution π, which is the limiting distribution. That is, for all j,
  $$\lim_{t\to\infty} P_{ij}(t) = \pi_j, \text{ for all initial } i.$$

- **Stationary distribution and generator matrix**
  A probability distribution π is a stationary distribution of a continuous-time Markov chain with generator Q if and only if
$$\pi Q = 0.$$

Example 1.1 (Continued) Find the stationary distribution.
The stationary distribution is the solution to $\pi Q = 0$.

$$\pi \begin{pmatrix} -2 & 1 & 1 \\ 2 & -5 & 3 \\ 3 & 1 & -4 \end{pmatrix} = 0,$$

that is

$$\pi = \begin{pmatrix} 17/30 & 1/6 & 4/15 \end{pmatrix}$$

3 Problems

1. **This problem explores the relationship between stationary distributions of a continuous-time Markov chain and its embedded chain.**
Assume that (X_t) is a continuous time Markov chain on state space S with generator matrix Q and embedded chain transition matrix $\tilde{P}$. We have learned that the stationary distribution π of the Markov chain is the solution to $\pi Q = 0$. However, for a continuous-time Markov chain, the stationary distribution π is not the same as the stationary distribution of the embedded chain.
Let ψ be a distribution on S with

$$\psi_j = \frac{\pi_j q_j}{\sum_k \pi_k q_k}$$

(a) Show ψ is a stationary distribution of the embedded chain.

Solution. To show ψ is a stationary distribution of the embedded chain. We just need to show that

$$\psi \tilde{P} = \psi.$$

The i^{th} entry of $\psi \tilde{P}$ is

$$\begin{aligned}
(\psi \tilde{P})_i &= \sum_j \psi_j \tilde{P}_{ji} \\
&= \sum_j \frac{\pi_j q_j}{\sum_k \pi_k q_k} p_{ji} \\
&= \frac{\sum_j \pi_j q_j p_{ji}}{\sum_k \pi_k q_k} \\
&= \frac{\pi_i q_i p_{ii} + \sum_{j \neq i} \pi_j q_{ji}}{\sum_k \pi_k q_k} \\
&= \frac{\sum_{j \neq i} \pi_j q_{ji}}{\sum_k \pi_k q_k} \text{ since } p_{ii} = 0 \\
&= \frac{\pi_i q_i}{\sum_k \pi_k q_k} \text{ since } \pi Q = 0 \\
&= \psi_i
\end{aligned}$$

(b) In part(a), we have shown that the stationary distribution of embedded chain ψ can be derived from π. Show that we can also derive π from ψ, with

$$\pi_j = \frac{\psi_j / q_j}{\sum_k \psi_k / q_k}$$

Solution. Since $\psi_j = C \pi_j q_j$, where C is an appropriate normalizing constant, we have that $\pi_j = \psi_j / (C q_j)$. This shows that

$$\pi_j \propto \frac{\psi_j}{q_j},$$

which gives

$$\pi_j = \frac{\psi_j / q_j}{\sum_k \psi_k / q_k}$$

(c) Suppose the embedded chain of a continuous-time process with states $\{1, 2, 3\}$ has transition matrix

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{pmatrix}.$$

Assume that the process stays at state 1 on average 5 minutes before moving to 2. From state 2, it stays on average 2 minutes before moving to a new state. From 3, it stays on average 4 minutes before transitioning to 2. Find the stationary distribution of the continuous-time chain.

Solution1. Give $\tilde{P}$ and holding time parameters, we can write out the generator matrix Q , and solve $\pi Q = 0$.

Solution2. We can also use the relationship between π and ψ obtained in the previous part. $\psi\tilde{P} = \psi$ gives $\psi = (1/6, 1/2, 1/3)$. Holding time parameters are $(q_1, q_2, q_3) = (1/5, 1/2, 1/4)$. The stationary distribution π is proportional to

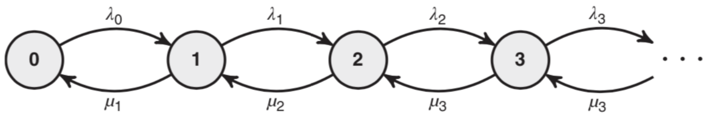
$$(\psi_1/q_1, \psi_2/q_2, \psi_3/q_3) = (5/6, 2/2, 4/3),$$

after normalization, this gives $\pi = (5/19, 6/19, 8/19)$

(d) **How should we interpret π and ψ?**

Solution. There is a difference in interpretation of the two distributions. The stationary probability π_j is the long-term proportion of time that the process spends in state j. On the other hand, the embedded chain stationary probability ψ_j is the long-term proportion of transitions that the process makes into state j.

2. (Birth-and-Death process) A birth-and-death Markov chain is a process with countably infinite state space $\{0, 1, \dots\}$ and two types of transitions: births from i to $i+1$ and deaths from i to $i-1$. Assume births occur from i to $i+1$ at the rate λ_i, and deaths occur from i to $i-1$ at the rate μ_i.



(a) Find the generator matrix.

Solution.

$$Q = \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

(b) Show $\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} < \infty$ is a necessary and sufficient condition for the stationary distribution to exist. And find the stationary distribution.

Solution. To find the stationary distribution, we need to solve $\pi Q = 0$, which is the system of linear equations:

$$\begin{aligned} -\pi_0\lambda_0 + \lambda_1\mu_1 &= 0 \\ \pi_0\lambda_0 - \pi_1(\lambda_1+\mu_1) + \pi_2\mu_2 &= 0 \\ &\vdots \end{aligned}$$

Solving the equations gives

$$\begin{aligned} \pi_1 &= \pi_0\frac{\lambda_0}{\mu_1} \\ \pi_2 &= \pi_1\frac{\lambda_1}{\mu_2} = \pi_0\frac{\lambda_0\lambda_1}{\mu_1\mu_2} \\ &\vdots \\ \pi_k &= \pi_0\frac{\pi_0 \dots \pi_{k-1}}{\mu_1 \dots \dots \mu_k} = \pi_0\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i} \end{aligned}$$

To make the components of π sum up to 1, we need

$$\sum_{k=0}^{\infty} \pi_k = \pi_0 \sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i} = 1.$$

Hence a necessary and sufficient condition for the stationary distribution to exist is

$$\sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i}<\infty.$$

Then we can solve for π_0 and obtain the stationary distribution π, that is

$$\pi_0 = (\sum_{k=0}^{\infty}\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i})^{-1}$$

$$\pi_k = \pi_0\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i} \text{ for } k = 1, 2, \ldots$$