Section 8 (Stat 171)

- TF: Jiaze Qiu
  - Email: jiazeqiug.harvard.edu
  - Section/OH: Wed 9-12 pm (ET)
- TF: Fernando Vicente
  - Email: fernando_vicente@fas.harvard.edu
  - Section/OH: Mon 7-9 pm and Thu 8-10 pm (ET)

- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

1 Review

Continuous-time Markov Chains (CTMC)
1. Three equivalent definitions of continuous-time Markov chains

1. Transition function definition
   - A continuous-time stochastic process $(X_t)_{t\geq 0}$ with discrete state space S is a continuous-time Markov chain if it satisfies the Markov property:
   $$P(X_{t+s} = j|X_s = i, X_u = x_u, 0 \leq u \leq s) = P(X_{t+s} = j|X_s = i),$$
   for all $s, t \geq 0; i, j, x_u \in S$.
   - A process is time-homogeneous if the probability does not depend on s:
   $$P(X_{t+s} = j|X_s = i) = P(X_t = j|X_0 = i).$$
   - So far, the only non-time-homogeneous process you should know is the non-homogeneous Poisson process.
   - For $t \geq 0$, the transition function $P(t)$ is a matrix function with
   $$P_{ij}(t) = P(X_t = j|X_0 = i).$$
   - Similar to the transition matrix of discrete-time Markov chain, the transition function also satisfies the Chapman-Kolmogorov equations:
   $$P(s+t) = P(s)P(t),$$
   which means
   $$P_{ij}(s+t) = \sum_k P_{ik}(s)Pkj(t).$$
2. Holding times and embedded chains definition A continuous-time Markov chain can be described by specifying holding times and embedded chain.
   - The holding time at state i, denoted as T_i, is the length of time that a continuous-time Markov chain started in i stays in i before transitioning to a new state.
   - T_i is memoryless and a real-valued function, thus follows an exponential distribution. Let q_i be its parameter, i.e., $T_i \sim Expo(q_i)$
   - If $q_i = 0$, the process never leaves i once it hits i, so i is an absorbing state
   - If $q_i = \infty$, the process leaves i immediately upon entering i, which allows for infinitely many transitions in a finite interval, so this process is called explosive.

- If we ignore time and look at state to state transitions, then we see a sequence of states $Y_0, Y_1, \ldots$, where Y_n is the nth state visited by the continuous process. The sequence $Y_0, Y_1, \ldots$ is a discrete-time Markov Chain called the embedded chain.
- Denote $\tilde{P} = (p_{ij})$ be the transition matrix of the embedded chain. Note that the diagonals entries of $\tilde{P}$ are 0.
- We can simulate the continuous-time Markov chain by waiting a time T_i, and then jumping to the next state according to p_{ij}. The process now waits a time T_{j^*} and chooses the next state according to p_{j^*k}. If time is not of interest, we could disregard the holding times and only simulate the jumps.

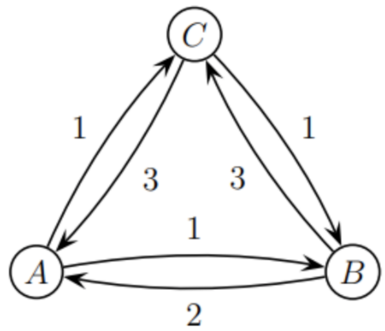
3. Transition rates (alarm clocks)
   - A continuous Markov chain also be specified by transition rates between pairs of state, as we can obtain the holding time parameters and the embedded chain transition probabilities from the transition rates.
   - Let q_{ik} be the transition rates.
   - Starting from state i, for each state k that the process can visit from i, suppose there is an alarm clocks associated with (i, k) that will ring after length of time T_{ik}, which is exponentially distributed with parameter q_{ik}. The clocks are independent started simultaneously. If the (i, j) clock rings first, then the process moves to state j.
   - The holding time $T_i = \min_k(T_{ik}) \sim Expo(\sum_k q_{ik})$. From the discussion of holding times, the exponential distribution parameter of T_i is q_i. It follows that
$$q_i = \sum_k q_{ik}.$$
   - The transition proability of the embedded chain is
$$p_{ij} = P(T_i = T_{ij}) \frac{q_{ij}}{\sum_k q_{ik}} = \frac{q_{ij}}{q_i}.$$
   - The infinitesimal generator Q characterizes the continuous-time Markov process
   - $Q_{ij} = q_{ij}$ for $i \neq j$ and $Q_{ii} = -\sum_{j \neq i} q_{ij} = -q_i$, where $q_i j$ are the transition rates and q_i is the holding time.
   - The generator matrix is **not** a stochastic matrix: the diagonals entries are negative, and the rows sum to 0.
   - The transition function $P(t)$ can be derived from Q. The forward and backward Kolmogorov equation state
$$P'(t) = P(t)Q$$
and
$$P'(t) = QP(t),$$
respectively. The solution to this equation is analogous to its one-dimensional form: $P(t) = e^{tQ} \equiv I + tQ + \frac{t^2}{2!}Q^2 + \frac{t^3}{3!}Q^3 + \cdots$.
   - If Q is diagonalizable, i.e. $Q = SD^{-1}S$, we get a more tractable solution
$$P(t) = e^{tQ} = Se^{tD}S^{-1}$$
.

2. Long-term behavior

- Limiting distribution:
$$\lim_{t\infty} P_{ij}(t) = \pi_j.$$
for all $i, j \in S$.
- Stationary distribution:
$$\pi = \pi P(t)$$
for all $t \geq 0$.
- As in the discrete case, the limiting distribution of a continuous-time Markov chain is a stationary distribution, but the converse is not necessarily true.
- The notions of accessibility, communication, and irreducibility are defined as in the discrete case, with one difference: for a continuous-time Markov chain, all states are aperiodic.
- Fundamental Limit Theorem: Let $(X_t)_{t \geq 0}$ be a finite, irreducible, continuous-time Markov chain with transition function $P(t)$. Then, there exists a unique stationary distribution π, which is the limiting distribution.
- A probability distribution π is a stationary distribution of a continuous-time Markov chain with generator Q if and only if $\pi Q = 0$.

2 Problems

1. For the Markov chain with transition rate graph shown below:



(a) Find the generator matrix.

(b) Find the transition matrix of the embedded chain.

(c) Find the stationary distribution.

2. Relationship between stationary distribution of CTMC and embedded chain. Let the CTMC be described by holding time (q_k). Let ψ be a distribution on the state space S with $\psi_j = \frac{\pi_j q_j}{\sum_k \pi_k q_k}$, where pi_j denote the stationary distribution of the CTMC.

(a) Show that ψ is a stationary distribution of the embedded chain.

(b) Show that we can also derive the CTMC stationary distribution π from ψ, with $\pi_j = \frac{\psi_j / q_j}{\sum_k \psi_k / q_k}$.

(c) Suppose the embedded chain of a CTMC with states $\{1, 2, 3\}$ is

$$\tilde{P} = \begin{pmatrix} 0 & 1 & 0 \\ 1/3 & 0 & 2/3 \\ 0 & 1 & 0 \end{pmatrix}$$

Assume that the process stays at state 1 on average 5 minutes before moving to 2. From state 2, it stays on average 2 minutes before moving to a new state. From 3, it stays on average 4 minutes before transitioning to 2. Find the stationary distribution of the continuous-time chain.

(d) How should we interpret π and ψ?

3. Birth-and-Death process. A birth-and-death Markov chain is a process with countably infinite state space $\{0, 1, \ldots\}$ and two types of transitions: births from i to $i+1$ and deaths from i to $i-1$. Assume births occur from i to $i+1$ at the rate λi , and deaths occur from i to $i-1$ at the rate μi. All other transitions have rates 0.
(a) Find the generator matrix.

(b) Show that $\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} < \infty$ is a necessary and sufficient condition for the stationary distribution to exist. And find the stationary distribution.