2 Problems

1. In a branching process the number of offspring per individual has a binomial distribution with parameters 2, p. Starting with a single individual, calculate:

   (a) the extinction probability;

   Solution. First calculate the generating function G.

   $$G(s) = (1-p)^2 + 2p(1-p)s + p^2s^2. \tag{1}$$

   Solving $G(s) = s$ gives

   $$\begin{cases} s_1 = 1, \\ s_2 = \dfrac{(1-p)^2}{p^2}. \end{cases} \tag{2}$$

   Thus, the extinction probability is 1 if $p \le 1/2$ and $\frac{(1-p)^2}{p^2}$ otherwise.

   (b) the probability that the population becomes extinct for the first time in the second generation.

   Solution. Let $\{Z_n\}$ denote the process.

   $$\begin{aligned} &P(Z_2 = 0 \text{ for the first time}|Z_0 = 1) \\ =&P(Z_2 = 0, Z_1 = 1|Z_0 = 1) + P(Z_2 = 0, Z_1 = 2|Z_0 = 1) \\ =&P(Z_2 = 0|Z_1 = 1)P(Z_1 = 1|Z_0 = 1) + P(Z_2 = 0|Z_1 = 2)P(Z_1 = 2|Z_0 = 1) \\ =&(1-p)^2 \cdot 2p(1-p) + (1-p)^4 \cdot p^2 \\ =&p(1-p)^3(2+p-p^2). \end{aligned} \tag{3}$$

   (c) Suppose that, instead of starting with a single individual, the initial population size is a random variable that is Poisson distributed with mean λ. Show that, in this case, the extinction probability is given, for $p > 1/2$, by

   $$\exp\left\{\lambda(1-2p)/p^2\right\}. \tag{4}$$

   Proof. When $p > 1/2$, the extinction probability for one individual is given in (a) as $\pi = \frac{(1-p)^2}{p^2}$. By conditioning on the initial population size,

   $$\begin{aligned} P(\text{extinction}) &= \sum_{k=0}^{\infty} P(\text{extinction}|Z_0 = k)P(Z_0 = k) \\ &= \sum_{k=0}^{\infty} \pi^k \frac{\lambda^k}{k!} e^{-\lambda} \\ &= \exp\{\pi\lambda - \lambda\} \\ &= \exp\left\{\lambda(1-2p)/p^2\right\}. \end{aligned} \tag{5}$$

2. Consider a branching process in which the number of offspring per individual has a Poisson distribution with mean λ, $\lambda > 1$. Let π_0 denote the probability that, starting with a single individual, the population eventually becomes extinct. Also, let a, $a < 1$, be such that

$$ae^{-a} = \lambda e^{-\lambda}. \tag{6}$$

(a) Show that $a = \lambda \pi_0$.

Proof. Compute $G(s)$ first.

$$G(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{\lambda(s-1)}. \tag{7}$$

The function $f(s) = G(s) - s$ is convex (positive second derivative), so it has at most 2 roots. 1 is an obvious root, and $f'(1) = \lambda - 1 > 0$, so there is another smaller root. Thus, $\pi_0 < 1$ is the only root other than 1. Plug in π_0, we have

$$\pi_0 = e^{\lambda(\pi_0 - 1)}, \tag{8}$$

which is equivalent to

$$\lambda \pi_0 e^{-\lambda \pi_0} = \lambda e^{-\lambda}. \tag{9}$$

Hence it's clear that a/λ is a root of f. Since $a/\lambda \neq 1$ (otherwise $a = \lambda > 1$), it has to be π_0.

(b) Show that, conditional on eventual extinction, the branching process has the same transition probabilities as the branching process in which the number of offspring per individual is Poisson with mean a.

Proof. Let Z_n be the original process (with parameter λ), and let X_n be the process with parameter a. Conditional on $Z_n = X_n = k$, let the k individuals in the two processes be $\{S_i\}_{i=1}^k$ and $\{T_j\}_{j=1}^k$ respectively. We prove the result if we can show that for any $(n_1, n_2, \ldots, n_k)$,

$$\begin{aligned} &P(S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k | \text{eventual extinction}, Z_n = k) \\ =&P(T_j \text{ has } n_j \text{ offspring}, j = 1, \ldots, k | X_n = k). \end{aligned} \tag{10}$$

The right hand side is just $\prod_{j=1}^k \frac{a^{n_j}}{n_j!} e^{-a} = e^{-ak} \prod_{j=1}^k \frac{a^{n_j}}{n_j!}$. Now we find the left hand side.

$$\begin{aligned}
& P(S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k | \text{eventual extinction}, Z_n = k) \\
=& \frac{P(S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k, \text{eventual extinction} | Z_n = k)}{P(\text{eventual extinction} | Z_n = k)} \\
=& \frac{P(S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k | Z_n = k) P(\text{e.e.} | S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k, Z_n = k)}{P(\text{eventual extinction} | Z_n = k)} \\
=& \frac{P(S_i \text{ has } n_i \text{ offspring}, i = 1, \ldots, k | Z_n = k) P(\text{e.e.} | Z_{n+1} = \sum_{i=1}^k n_i)}{P(\text{eventual extinction} | Z_n = k)} \\
=& \frac{\prod_{i=1}^k \frac{\lambda^{n_i}}{n_i!} e^{-\lambda} \pi_0^{\sum_{i=1}^k n_i}}{\pi_0^k} \\
=& \frac{\prod_{i=1}^k \frac{(\lambda \pi_0)^{n_i}}{n_i!} e^{-\lambda}}{\pi_0^k} \\
=& \prod_{i=1}^k \frac{(\lambda \pi_0)^{n_i}}{n_i!} \frac{e^{-\lambda}}{\pi_0} \\
=& \prod_{i=1}^k \frac{(\lambda \pi_0)^{n_i}}{n_i!} e^{-\lambda \pi_0} \text{ (see (9))} \\
=& \prod_{i=1}^k \frac{a^{n_i}}{n_i!} e^{-a} = e^{-ak} \prod_{i=1}^k \frac{a^{n_i}}{n_i!}.
\end{aligned} \tag{11}$$