Section 14 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 The Ito Integral

1.1 Definition

The Ito integral is defined as

$$\int_0^t X_s dB_s = \lim_{n\to\infty} \sum_{k=1}^{n} X_{t_{k-1}}(B_{t_k} - B_{t_{k-1}}), \tag{1}$$

for ever-finer partitions $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$.

Requirements:

1. $\int_0^t E(X_s^2)ds < \infty$

2. X_t does not depend on $\{B_s : s > t\}$ but can depend on previous s $\{B_s : s \leq t\}$ (X_t is adapted to brownian motion)

3. $\lim_{n\to\infty} \sum_{k=1}^{n} X_{t_{k-1}}(B_{t_k} - B_{t_{k-1}})$ converges in a mean square sense to $\int_0^t X_s dB_s$

1.2 Ito's Lemma

Ito's Lemma

Ito's lemma gives us a term for $dg(B_t)$ using the Taylor Expansion formula. Let g be a real valued, twice differentiable function. Then Ito's lemma is as follows:

$$g(B_t) - g(B_0) = \int_0^t g'(B_s)dB_s + \frac{1}{2}\int_0^t g''(B_s)ds$$

In shorthand differential form:

$$dg(B_t) = g'(B_t)dB_t + \frac{1}{2}g''(B_t)dt$$

(Extension of) Ito's Lemma

Let $g(t, x)$ be a real-valued function whose second-order partial derivatives are continuous. Then,

$$\begin{aligned} g(t, B_t) - g(0, B_0) = &\int_0^t \left(\frac{\partial}{\partial t}g(s, B_s) + \frac{1}{2}\frac{\partial^2}{\partial x^2}g(s, B_s)\right) ds \\ &+ \int_0^t \frac{\partial}{\partial x}g(s, B_s)dB_s. \end{aligned} \tag{2}$$

In shorthand differential form,

$$dg = \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\right) dt + \frac{\partial g}{\partial x}dB_t. \tag{3}$$

2 Stochastic Differential Equations

Recall the traditional differential equation:

- $\frac{dx}{dt} = f(t, x)$
- $X(0) = X_0$

We commonly see stochastic differential equations used when describing **diffusion**. The diffusion SDE is described as follows:

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dB_s$$

Where $\alpha(t, X_t)$ is a drift coefficient and $\beta(t, X_t)$ is the diffusion coefficient. We can rewrite this in integral form:

$$X_t - X_0 = \int_0^t \alpha(s, X_s)ds + \int_0^t \beta(s, X_s)ds$$

Notice this form is somewhat similar to Ito's Lemma. Let us introduce **Ito's Lemma for diffusion**:

Let g(t,x) be a real-valued function w/ continuous second-order derivatives. Let X_t be an Ito process such that:

$$dX_t = \alpha(t, X_t)dt + \beta(t, X_t)dB_t$$

As before. Then we have:

$$\begin{aligned} g(t, X_t) - g(0, X_0) = &\int_0^t \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\alpha(s, X_s) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\beta^2(s, X_s) \right) ds \\ &+ \int_0^t \frac{\partial g}{\partial x}\beta(s, X_s)dB_s. \end{aligned} \tag{4}$$

In shorthand differential form:

$$dg = (\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}\alpha(s, X_s) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\beta^2(s, X_s))dt + \frac{\partial g}{\partial x}\beta(t, X_t)dB_t.$$

Problems

1. Use Ito's Lemma to evaluate $d(B_t^4)$ and $E(B_t^4)$.

Solution. Let $g(t,x) = x^4$. Then by Ito's Lemma, we have

$$d(B_t^4) = 6B_t^2 dt + 4B_s^3 dB_t$$

in differential form and

$$B_t^4 = 6\int_0^t B_s^2 ds + 4\int_0^t B_s^3 dB_s$$

Taking expectation

$$\begin{aligned} E[B_t^4] &= 6\int_0^t E[B_s^2]ds + 4E[\int_0^t B_s^3 dB_s] \\ &= 6\int_0^t s ds \\ &= 3t^2 \end{aligned}$$

2. Use Ito's Lemma to show that $E[B_t^k] = \frac{k(k-1)}{2}\int_0^t E[B_s^{k-2}]ds$, for $k \geq 2$.

Solution. Let $g(x) = x^k$. By Ito's Lemma,

$$d(B_t^k) = \frac{k(k-1)}{2} B_t^{k-2} dt + k B_t^{k-1} dB_t$$

in differential form and

$$B_t^k = \frac{k(k-1)}{2}\int_0^t B_t^{k-2} dt + k\int_0^t B_t^{k-1} dB_t$$

in integral form. Then

$$\begin{aligned} E[B_t^k] &= \frac{k(k-1)}{2}\int_0^t E[B_t^{k-2}]dt + k\int_0^t E[B_t^{k-1}]dB_t \\ &= \frac{k(k-1)}{2}\int_0^t E[B_s^{k-2}]ds \end{aligned}$$

Note: This is a recursive formula for the moments of a Normal distribution with mean 0 and variance t.

3. (a) Evaluate $d(tB_t^2)$.

(b) Derive a martingale that is a fourth-degree polynomial function of Brownian motion. Hint: use the result in (a) and Problem 1.

Solution. (a) Let $g(t,x) = tx^2$, then

$$\frac{\partial g}{\partial t} = x^2, \frac{\partial g}{\partial x} = 2tx, \frac{\partial^2 g}{\partial x^2} = 2t.$$

By Ito's Lemma,

$$d(tB_t^2) = (B_t^2 + t)dt + 2tB_t dB_t.$$

(b) Form problem 1,

$$B_t^4 = 6\int_0^t B_s^2 ds + 4\int_0^t B_s^3 dB_s.$$

Now consider

$$\begin{aligned}
4\int_0^t (B_s^3 - 3sB_s)dB_s &= 4\int_0^t B_s^3 dB_s - 6\int_0^t 2sB_s dB_s \\
&= (B_t^4 - 6\int_0^t B_s^2 ds) - 6[tB_t^2 - \int_0^t (B_s^2 + s)ds] \\
&= B_t^4 - 6tB_t^2 + 6\int_0^t s ds \\
&= B_t^4 - 6tB_t^2 + 3t^2
\end{aligned}$$

Since Ito integrals are martingales, the process $(B_t^4 - 6tB_t^2 + 3t^3)_{t\geq 0}$ is a martingale.

Show that if a function $f(t,x)$ satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0$$

then $f(t, B_t)$ is a martingale.

Solution. From the extended Ito's Lemma

$$\begin{aligned} df &= \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \frac{\partial f}{\partial x} dB_t \\ &= \frac{\partial f}{\partial x} dB_t \end{aligned} \tag{5}$$

in differential form and

$$f(t, B_t) - f(0, B_0) = \int_0^t \frac{\partial}{\partial x} g(s, B_s) dB_s. \tag{6}$$

in integral form.

Since Ito integrals are martingales, the process $f(t, B_t)$ is a martingale.