Section 10 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Topics list for midterm 2

- *Please remember earlier topics
- Metropolis-Hastings
- Gibbs samplings
- Ergodic Thm for MCs
- Convergence to stationarity
- Burn-in
- Definition of Poisson Process (PP)
  - Poisson distribution definition
  - Exponential inter-arrival times definition
  - Infinitesimal arrivals definition
- Superposition of Independent PP
- Thinning of Independent PP
- Conditional Distribution of Arrival Times, Uniform Order Statistics
- Spatial Poisson Process (PP)
- Non-homogeneous PP
- Waiting time paradox
- Definition of Continuous-time Markov Chain (CTMC)
  - Some features: time-homogeneity, transition function, Chapman-Kolmogorov Equations, holding times,transition rates
- Infinitesimal generator
- Embedded Chain
- Stationary distribution of the CTMC
- Stationary distribution of embedded chain

- Limiting Distribution of CTMC
- Fundamental Limit Theorem for Finite State Space
- Global balance equations
- Time reversibility (local balance equations)
- Communication classes (same as embedded chain)
- Recurrence/Transience (based on embedded chain)
- Positive recurrence of CTMC
- Fundamental Matrix Thm (Expected time to absorption)
- Birth/Death Chains
- Queuing Theory
- Definition Definition
- Martingales with respect to other processes
- Optional Stopping Theorem
- Wald's Lemma

2 Problems

1. Metropolis-Hasting algorithm Show how to generate a Poisson random variable with parameter λ using Metropolis-Hasting. Use simple symmetric random walk as the proposal distribution.

Solution. Let $U \sim Unif(0,1)$ random variable.

Since the proposal chain is SSRW then $T_{ij} = \begin{cases} 1/2 & i = j = 0 \\ 1/2 & |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$, where we have used a non-reflecting boundary at 0.

The acceptance function is $a_{ij} = \frac{\pi_j T_{ji}}{\pi_i T_{ij}} = \frac{\pi_j}{\pi_i}$, since the proposal T is symmetric. Let $X_n = i$. Then $X_{n+1} = \begin{cases} j & U \leq a_{ij} \\ i & U > a_{ij} \end{cases}$.

Case 1: $X_n = 0$,
$j = 0$ with probability 1/2.
$X_{n+1} = j = 0$ if $U \leq \frac{\exp^{-\lambda}}{\exp^{-\lambda}} = 1$.
$j = 1$ with probability 1/2.
$X_{n+1} = \begin{cases} j = 1 & U \leq \frac{\lambda \exp^{-\lambda}}{\exp^{-\lambda}} = \lambda \\ X_n = 0 & \text{otherwise} \end{cases}$.
Case 2: $X_n = i > 0$,
$j = i + 1$ with probability 1/2.
$X_{n+1} = \begin{cases} 1 & U \leq \frac{\lambda^{i+1} \exp^{-\lambda} / (i+1)!}{\lambda^i \exp^{-\lambda} / (i)!} = \frac{\lambda}{i+1} \\ i & \text{otherwise} \end{cases}$.
$j = i - 1$ with probability 1/2.
$X_{n+1} = \begin{cases} j = i - 1 & U \leq \frac{\lambda^{i-1} \exp^{-\lambda} / (i-1)!}{\lambda^i \exp^{-\lambda} / (i)!} = \frac{i}{\lambda} \\ i & \text{otherwise} \end{cases}$.

2. Poisson Process Let A be a space of size $|A| = 10$. Let $B \subseteq A$, $|B| = 3$. Let the number of dots in space A appear at rate λ per unit size per unit time. Let $N_t(A)$ be a Poisson Process that tracks the number of dots in space A, $N_t(A) \sim Poisson(\lambda |A| t)$. Find the probability that the first dot occurs in subspace B.

Solution. Let $N_t(B)$, $N_t(A - B)$ count the dots in subspace B, $A - B$, respectively. These are two independent Poisson processes with rates $\lambda|B|$, $\lambda(|A| - |B|)$.

We are asking for the probability that $N_t(B) = 1$ before $N_t(A - B) = 1$. This is the same as asking for the probability of $X_B \sim Expo(\lambda|B|)$ smaller than $X_{A-B} \sim Expo(\lambda(|A| - |B|)$. This event occurs with probability $\frac{\lambda|B|}{\lambda|B| + \lambda(|A| - |B|)} = \frac{|B|}{|A|}$.

3. Definitions in CTMC Assume that π is the limiting distribution of a continuous-time chain. Show that π is a stationary distribution. (Hint: start with the forward equation.)

Solution.

The forward equation states that $P'_{ij}(t) = \sum_k P_{ik}(t)Q_{kj}$.

Since π is the limiting distribution, we know that $\pi_j = \lim_{t\to\infty} P_{ij}(t)$ for all i. Note that that the function $P_{ij}(t)$ converges to a constant, it follows that its derivative, $P'_{ij}(t)$ must converge to 0.

Using the forward equations with the discussed limits gives:

$$0 \lim_{t\to\infty} P'_{ij}(t) = \lim_{t\to\infty} \sum_k P_{ik}(t)Q_{kj} = \sum_k \pi_k Q_{kj}$$

.

Thus the limiting distribution is the stationary distribution for CTMC.

Alternative Solution

The stationary distribution can also be formulated as $\pi_j = \sum_k \pi_k P_{kj}(t)$ for all t.

$$\pi_j = \lim_{t'\to\infty} P_{ij}(t') = \lim_{t'\to\infty} \sum_k P_{ik}(t'-t)P_{kj}(t) = \sum_k \lim_{t'\to\infty} \big(P_{ik}(t'-t)\big)P_{kj}(t) = \sum_k \pi_k P_{kj}(t)$$

4. Martingales Let $X_1, X_2, \ldots$ be i.i.d. random variables with $\mu < \infty$. Let $Z_n = \sum_{i=1}^{n}(X_i - \mu)$, for $n = 0, 1, 2, \ldots$ a) Show that $Z_0, Z_1, \ldots$ is a martingale with respect to $X_0, X_1, \ldots$.
Solution.

$$\begin{aligned} E(Z_{n+1}|\{X_i\}_1^n) &= E(\sum_{i=1}^{n+1}(X_i - \mu)|\{X_i\}_1^n) \\ &= \sum_{i=1}^{n}(X_i - \mu)E(X_{n+1} - \mu)|\{X_i\}_1^n) \\ &= Z_n \end{aligned}$$

b) Assume that N is a stopping time that satisfies the conditions of the optinal stopping theorem. Show that

$$E\left(\sum_{i=1}^{N} X_i\right) = E(N)\mu$$

.
Solution. By the optional stopping theorem, $E(Z_N) = E(Z_1) = 0$. It follows that

$$E\left(\sum_{i=1}^{N}(X_i - \mu)\right) = 0$$

$$E\left(\sum_{i=1}^{N} X_i\right) = E\left(\sum_{i=1}^{N} \mu\right) = E\left(N\mu\right) = E(N)\mu$$