Section 10 (Stat 171)

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- Acknolegement: This handout is partially based on notes created by Lisa Ruan and Christy Huo.

- All the section materials (handouts & solutions) can be found either on Canvas or here.

1 Topics list for midterm 2

- *Please remember earlier topics
- Metropolis-Hastings
- Gibbs samplings
- Ergodic Thm for MCs
- Convergence to stationarity
- Burn-in
- Definition of Poisson Process (PP)
  - Poisson distribution definition
  - Exponential inter-arrival times definition
  - Infinitesimal arrivals definition
- Superposition of Independent PP
- Thinning of Independent PP
- Conditional Distribution of Arrival Times, Uniform Order Statistics
- Spatial Poisson Process (PP)
- Non-homogeneous PP
- Waiting time paradox
- Definition of Continuous-time Markov Chain (CTMC)
  - Some features: time-homogeneity, transition function, Chapman-Kolmogorov Equations, holding times,transition rates
- Infinitesimal generator
- Embedded Chain
- Stationary distribution of the CTMC
- Stationary distribution of embedded chain

- Limiting Distribution of CTMC
- Fundamental Limit Theorem for Finite State Space
- Global balance equations
- Time reversibility (local balance equations)
- Communication classes (same as embedded chain)
- Recurrence/Transience (based on embedded chain)
- Positive recurrence of CTMC
- Fundamental Matrix Thm (Expected time to absorption)
- Birth/Death Chains
- Queuing Theory
- Definition Definition
- Martingales with respect to other processes
- Optional Stopping Theorem
- Wald's Lemma

2 Problems

1. Metropolis-Hasting algorithm Show how to generate a Poisson random variable with parameter λ using Metropolis-Hasting. Use simple symmetric random walk as the proposal distribution.

2. Poisson Process Let A be a space of size $|A| = 10$. Let $B \subseteq A$, $|B| = 3$. Let the number of dots in space A appear at rate λ per unit size per unit time. Let $N_t(A)$ be a Poisson Process that tracks the number of dots in space A, $N_t(A) \sim Poisson(\lambda|A|t)$. Find the probability that the first dot occurs in subspace B.

3. Definitions in CTMC Assume that π is the limiting distribution of a continuous-time chain. Show that π is a stationary distribution. (Hint: start with the forward equation.)

4. Martingales Let $X_1, X_2, \dots$ be i.i.d. random variables with $\mu < \infty$. Let $Z_n = \sum_{i=1}^{n}(X_i - \mu)$, for $n = 0, 1, 2, \dots$ a) Show that $Z_0, Z_1, \dots$ is a martingale with respect to $X_0, X_1, \dots$.
b) Assume that N is a stopping time that satisfies the conditions of the optinal stopping theorem. Show that

$$E\left(\sum_{i=1}^{N} X_i\right) = E(N)\mu$$

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